A $3D$ Semi implicit method for computing current density in bulk superconductors

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***Abstract*— A semi-implicit approach is proposed for computing current density in superconductors characterized by a non linear vectorial power law $\overrightarrow{J}(\overrightarrow{E}) \sim \|\overrightarrow{E}\|^{\frac{1}{n}-1}\overrightarrow{E}$, with $n > 1$. A Discontinuous Galerkin method is adopted for the spatial discretization of the non linear system satisfied by the components of $\overrightarrow{E}$ and $\overrightarrow{J}$. An application example is given for a superconducting cube subjected to an external magnetic field along the $z-$axis. The distributions of components J_x, J_y are symetric and reflect similarities of their boundaries conditions. The penetration time of the sample is compared to that obtained when a surounding vacuum volume is added on the domain.**

***Index Terms*— Superconductors, Nonlinear diffusion, Discontinuous Galerkin method, Newton iterative method, Semi-implicit formulation.**

I. Introduction

Development of numerical experiments on high temperature superconducting materials is a crucial issue for modeling their applications. This remains arduous notably when their behaviour is described by non linear relations $J-E$ or $E-J$ given by:

$$\frac{\overrightarrow{J}}{J_c} = \left\|\frac{\overrightarrow{E}}{E_c}\right\|^{\frac{1}{n}-1}\frac{\overrightarrow{E}}{E_c} \quad \text{or} \quad \frac{\overrightarrow{E}}{E_c} = \left\|\frac{\overrightarrow{J}}{J_c}\right\|^{n-1}\frac{\overrightarrow{J}}{J_c}, \tag{1}$$

where $\overrightarrow{J}$ is the current density, $\overrightarrow{E}$ the electric field, E_c the critical electric field, J_c the critical current density and $n > 1$ the power law exponent.

Different discretization techniques have been implemented for computing $2D$ induced fields in bulk superconductors when a non linear scalar problem is assumed [1], [2], [3]. The cases of $3D$ models are less common. A surrounding vacuum volume is always added around the superconducting domain in order to describe demagnetizing effects due to the induced current density [4], [5]. Unfortunately, this leads to resolving additional equations in the vacuum and makes it harder to carry on simulations with arbitrary geometries and shapes.

In this paper, a discontinuous Galerkin discretization approach is considered and a semi-implicit scheme is proposed for solving the non linear system formed by the components of the electric field. The boundary conditions are defined with an explicit evaluation of the demagnetizing field created by the sample. This work is an extension of the method published previously to compute the vectorial $2D$ current density distribution in superconductors [6].

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II. The differential system

In a three-dimensional setting where the magnetic induction is $\overrightarrow{B} = (B_x, B_y, B_z)$, the electric field and current density have three nonzero components. They satisfy a vectorial problem written as:

$$\frac{\partial\overrightarrow{J}}{\partial t} - \frac{1}{c}\overrightarrow{\triangle}\overrightarrow{E} = -\overrightarrow{\nabla}\overrightarrow{\nabla}\cdot\overrightarrow{E}, \tag{2}$$

with $c = \mu_0 J_c/E_c$.

The main idea of this article is to solve the diffusion equations of each component with an explicit coupling. We set $u_1 = E_x/E_c$, $u_2 = E_y/E_c$, $u_3 = E_z/E_c$, $(S_1, S_2, S_3) = -\overrightarrow{\nabla}\overrightarrow{\nabla}\cdot\overrightarrow{E}$ and define the following functions :

$$\beta_1(u_1,u_2,u_3) = \left(u_1^2+u_2^2+u_3^2\right)^{\frac{1-n}{2n}} u_1 = J_x/J_c = v_1 \tag{3}$$

$$\beta_2(u_1,u_2,u_3) = \left(u_1^2+u_2^2+u_3^2\right)^{\frac{1-n}{2n}} u_2 = J_y/J_c = v_2 \tag{4}$$

$$\beta_3(u_1,u_2,u_3) = \left(u_1^2+u_2^2+u_3^2\right)^{\frac{1-n}{2n}} u_3 = J_z/J_c = v_3 \tag{5}$$

The vectorial equation (2) forms a system of three scalar non linear diffusion equations. to which boundary conditions or the fluxes must be added (see Section IV). We perform a semi-implicit approach on the following system :

$$(S)\left\{\begin{array}{l} \dfrac{\partial\beta_1(u_1,u_2,u_3)}{\partial t} - c^{-1}\triangle u_1 = S_1 \text{ in } \quad \Omega \\ \dfrac{\partial\beta_2(u_1,u_2,u_3)}{\partial t} - c^{-1}\triangle u_2 = S_2 \text{ in } \quad \Omega \\ \dfrac{\partial\beta_3(u_1,u_2,u_3)}{\partial t} - c^{-1}\triangle u_3 = S_3 \text{ in } \quad \Omega \\ \overrightarrow{\nabla}u_1\cdot\overrightarrow{\nu} = C_{b_1}(t) \text{ on } \quad \partial\Omega \\ \overrightarrow{\nabla}u_2\cdot\overrightarrow{\nu} = C_{b_2}(t) \text{ on } \quad \partial\Omega \\ \overrightarrow{\nabla}u_3\cdot\overrightarrow{\nu} = C_{b_3}(t) \text{ on } \quad \partial\Omega \end{array}\right. . \tag{6}$$

where $C_{b_{1,2,3}}$ are built from Faraday law and assumption $\overrightarrow{\nabla}\cdot\overrightarrow{E} = 0$ on $\partial\Omega$.

III. Semi-implicit approach

We consider the time step $\delta t = t_{k+1} - t_k$. At time t_{k+1}, explicit forms u_2^k, u_3^k are used for determining the solution u_1^{k+1} (similarly for u_2^{k+1} and u_3^{k+1}). This assumption leads on these semi-implicit expressions of the nonlinearities:

$$\beta_1^{k+1} = \left[\left(u_1^{k+1}\right)^2 + \left(u_2^k\right)^2 + \left(u_3^k\right)^2\right]^{\frac{1-n}{2n}} u_1^{k+1} \tag{7}$$

$$\beta_2^{k+1} = \left[\left(u_1^{k}\right)^2 + \left(u_2^{k+1}\right)^2 + \left(u_3^k\right)^2\right]^{\frac{1-n}{2n}} u_2^{k+1} \tag{8}$$

$$\beta_3^{k+1} = \left[\left(u_1^{k}\right)^2 + \left(u_2^k\right)^2 + \left(u_3^{k+1}\right)^2\right]^{\frac{1-n}{2n}} u_2^{k+1} \tag{9}$$

The spatial discretization of differential operators in the system (6) are performed with a Discontinuous Galerkin method described in [6]. The generated terms are represented by $F_{1,2,3}$. The discrete system to solve is written :

$$\begin{cases} M\dfrac{\beta_1^{k+1} - \beta_1^k}{\delta t} = F_1(u_1^{k+1}, u_2^k, u_3^k) \\ M\dfrac{\beta_2^{k+1} - \beta_2^k}{\delta t} = F_2(u_1^k, u_2^{k+1}, u_3^k) \\ M\dfrac{\beta_3^{k+1} - \beta_3^k}{\delta t} = F_3(u_1^k, u_2^k, u_3^{k+1}) \end{cases} \tag{10}$$

To avoid difficulties in the linearization process due to the fact that β_1, β_2 and β_3 are not derivable at 0, the discrete unknowns are changed in (10). The components of current density v_1, v_2, v_3 are calculated from an equivalent discrete system given by:

$$\begin{cases} M\dfrac{v_1^{k+1} - v_1^k}{\delta t} = G_1(v_1^{k+1}, v_2^k, v_3^k) \\ M\dfrac{v_2^{k+1} - v_2^k}{\delta t} = G_2(v_1^k, v_2^{k+1}, v_3^k) \\ M\dfrac{v_3^{k+1} - v_3^k}{\delta t} = G_3(v_1^k, v_2^k, v_3^{k+1}) \end{cases} \tag{11}$$

where $G(v_1, v_2, v_3) = F(u_1, u_2, u_3)$. The linearization of G is ensured if the inverse functions β_1^{-1} β_2^{-1} β_3^{-1} exist and are derivable. Unfortunately, these derivatives are non trivial. In accordance with the $E - J$ power law (1), we suppose that :

$$u_1^{k+1} = \left[\left(v_1^{k+1}\right)^2 + \left(v_2^k\right)^2 + \left(v_3^k\right)^2\right]^{\frac{n-1}{2}} v_1^{k+1} \tag{12}$$

$$u_2^{k+1} = \left[\left(v_1^{k}\right)^2 + \left(v_2^{k+1}\right)^2 + \left(v_3^k\right)^2\right]^{\frac{n-1}{2}} v_2^{k+1} \tag{13}$$

$$u_3^{k+1} = \left[\left(v_1^{k}\right)^2 + \left(v_2^k\right)^2 + \left(v_3^{k+1}\right)^2\right]^{\frac{n-1}{2}} v_3^{k+1} \tag{14}$$

These inverse functions are continuous and derivable. A Newton iterative method is applied for computing v_1^{k+1}, v_2^{k+1} and v_3^{k+1}.

IV. Boundary conditions

For getting the fluxes at the border, components of $\overrightarrow{\nabla} u_{1,2,3}$ are expressed in an explicit form. For example, for u_1:

$$\begin{cases} \overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \\ \overrightarrow{rot}\,\overrightarrow{E} = -\dfrac{\partial \overrightarrow{B}}{\partial t} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial u_1^{k+1}}{\partial x} = -\dfrac{\partial u_2^k}{\partial y} - \dfrac{\partial u_3^k}{\partial z} \\ \dfrac{\partial u_1^{k+1}}{\partial y} = \dfrac{\partial u_2^k}{\partial x} + \dfrac{\partial B_z}{\partial t} \\ \dfrac{\partial u_1^{k+1}}{\partial z} = \dfrac{\partial u_3^k}{\partial x} - \dfrac{\partial B_y}{\partial t} \end{cases} \tag{15}$$

where $\overrightarrow{B} = \overrightarrow{B_a} + \overrightarrow{B_d}$, with $\overrightarrow{B_a}$ the uniform applied magnetic field and $\overrightarrow{B_d}$ the demagnetizing field created by the sample. It is well known that the average magnetic induction around the sample is given by $\overrightarrow{B} = \mu_0(\overrightarrow{H_a} + \overrightarrow{M})$, where M is magnetization of the sample defined by $M = 0.5V \int_V \overrightarrow{r} \times \overrightarrow{J} dV$, with $\overrightarrow{r} = (x, y, z)$ and V the volume.
We consider a quasi-static approximation of the demagnetizing field and suppose that $B_{d_{x,y,z}} = -\mu_0 N_{x,y,z} M_{x,y,z}$, where N is the demagnetizing factor in each directions. They are evaluated thanks to analytical formulas [7]. We will develop expressions of boundaries conditions while taking into account the demagnetizing field in the extended paper.

V. Numerical results

We consider a superconducting cube of edge length $a = 1mm$, characterized by $J_c = 50A/mm^2$, $E_c = 10^{-7}V/mm$ and $n = 20$. It is subjected to an external magnetic field in the z direction such as, $B_a(t) = a_B t$, with $a_B = 1T/s$.

The distributions J_x/J_c, J_y/J_c are presented in Fig.1 at $t_{PDG} = 0.03s$ when the full penetration is reached. They are symmetric and reflect similarity of their boundaries conditions. The penetration time t_{PDG} is close to that obtained with the Finite volume method applied on $A - V$ formulation with a surrounding vacuum volume $t_{PFV} = 0.033s$ [5]. These numerical results and comparison will be detailed in the extended paper.

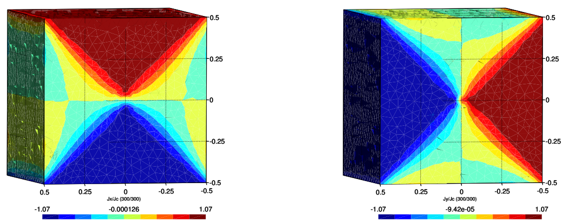

Fig. 1. J_x/J_c (left) and J_y/J_c (right) at $t = 0.03s$

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