A WENO Scheme for Streamer Discharge Simulations

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***Abstract*—This paper proposes a WENO finite difference method to simulate the fluid model of streamer discharges. To simulate the rapid transient streamer discharge process, a method with high resolution and high order accuracy is highly desired. High order WENO is such a choice. A simulation of a double-headed streamer discharge in Nitrogen was performed using 2-dimensional fluid model. The preliminary results indicate the potential of extending the method to general streamer simulations.**

I. Introduction

The streamer discharge is the initial stage of various electrical discharges that happens every day. Most micro streamer discharge parameters still cannot be measured by experiments, which makes numerical simulations essential tools for a better understanding of streamer physics.

The simplest and most frequently used model for streamer discharge is the fluid model, which consists of two continuity equations (which are convection-dominated diffusion equations with source terms) coupled with a Poisson's equation, see Eq. (1) to Eq. (5), where $n_{e,p}$ are the charged particle densities, $\mu_{e,p}$ are the movability coefficients, $v_{e,p}$ are the convection velocities, D_r and D_z are the diffusion coefficients, the index e, p means electrons, positive ions, respectively. U and E are the electrical potential and electric field, respectively; ε_0 is the dielectric coefficient in air; e_0 is the unit charge of an electron. α and η are measured by experiments. See [1] for details.

$$\frac{\partial n_e}{\partial t}+\frac{1}{r}\frac{\partial(rv_{er}n_e)}{\partial r}+\frac{\partial(v_{ez}n_e)}{\partial z}-\frac{D_r}{r}\frac{\partial}{\partial r}(r\frac{\partial n_e}{\partial r}) -D_z\frac{\partial^2 n_e}{\partial z^2}=\alpha n_e|v_e|, \quad (1)$$

$$\frac{\partial n_p}{\partial t}+\frac{1}{r}\frac{\partial(rv_{pr}n_p)}{\partial r}+\frac{\partial(v_{pz}n_p)}{\partial z}=\alpha n_e|v_e|, \quad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\varepsilon_0\frac{\partial U}{\partial r})+\frac{\partial}{\partial z}(\varepsilon_0\frac{\partial U}{\partial z})=e_0(n_e-n_p), \quad (3)$$

$$E=(E_r,E_z)^T=-(\frac{\partial U}{\partial r},\frac{\partial U}{\partial z})^T,\ |E|=\sqrt{E_r^2+E_z^2}, \quad (4)$$

$$v_{e,p}=(v_{(e,p)r},v_{(e,p)z})^T=\mu_{e,p}E,\ |v_e|=\sqrt{v_{er}^2+v_{ez}^2}. \quad (5)$$

The continuity equations, i.e., Eq. (1) and Eq. (2), are convection dominated. Godunov show that high order linear numerical schemes may generate numerical oscillations for these equations [2]. In addition, first order schemes for convection dominated problems suffer from numerical diffusions. A high order scheme free of numerical oscillations and of high resolution is greatly desired.

In this paper, we use a WENO finite difference scheme to solve the continuity equations due to its high numerical stability, high order accuracy and high efficiency.

II. A WENO scheme for continuity equation discretization

We take the governing equation of electrons for example. Multiplying Eq (1.1) by r, we get

$$\frac{\partial(rn_e)}{\partial t}+\frac{\partial(rv_{er}n_e)}{\partial r}+\frac{\partial(rv_{ez}n_e)}{\partial z}-\frac{\partial}{\partial r}(rD_r\frac{\partial n_e}{\partial r}) -\frac{\partial}{\partial z}(rD_z\frac{\partial n_e}{\partial z})=r(\alpha-\eta)n_e|v_e|. \quad (6)$$

A. Spacial discretization

To illustrate the main idea of the WENO finite difference scheme, we first consider the following 1D case:

$$\frac{\partial(ru)}{\partial t}+\frac{\partial\left(rf(u)\right)}{\partial r}=0. \quad (7)$$

Define $r_{i+\frac{1}{2}}=r_i+\frac{1}{2}\triangle r$, $r_{i-\frac{1}{2}}=r_i-\frac{1}{2}\triangle r$, the finite difference scheme for Eq. (7) is given

$$\frac{\mathrm{d}(r_iu_i)}{\mathrm{d}t}+\frac{1}{\triangle r}\left(\widehat{(rf)}_{r_{i+\frac{1}{2}}}-\widehat{(rf)}_{r_{i-\frac{1}{2}}}\right)=0. \quad (8)$$

Eq. (8) is k-th order accurate in space if

$$\frac{1}{\triangle r}\left(\widehat{(rf)}_{r_{i+\frac{1}{2}}}-\widehat{(rf)}_{r_{i-\frac{1}{2}}}\right)=\frac{\partial(rf)}{\partial r}|_{r=r_i}+O(\triangle r^k). \quad (9)$$

Eq. (9) can be achived by the WENO reconstruction [3]. Assuming $f'(u)\geq 0$, at time t^n,

1) Obtain the cell average of an implicit existing polynomial h on cell i by $\overline{h}_i^n=r_if(u_i)$;
2) Use WENO reconstruction based on $\overline{h}_i^n$ and $\overline{h}_j^n$, where cells j are in the neighborhood of cell i, to construct the point value at $r_{i+\frac{1}{2}}$, and denote it by $h^-_{i+\frac{1}{2}}$;
3) Set the flux $\widehat{(rf)}_{i+\frac{1}{2}}=h^-_{i+\frac{1}{2}}$;

On general occasions that $f(u)$ is not locally monotone over a stencil, the following flux splittings are performed:

$$p_+^{i+\frac{1}{2}}=\frac{1}{2}r(u+\frac{f(u)}{\alpha_{i+\frac{1}{2}}}),\quad p_-^{i+\frac{1}{2}}=\frac{1}{2}r(u-\frac{f(u)}{\alpha_{i+\frac{1}{2}}}), \quad (10)$$

where $\alpha_{i+\frac{1}{2}}$ is the local maximal of $|f'(u)|$ over the stencil, and the superscript $i+\frac{1}{2}$ is used to clarify that the splitting is related to the interface $r_{i+\frac{1}{2}}$. At time level t^n, for each fixed

interface $r_{i+\frac{1}{2}}$, the procedure to reconstruct the flux $\widehat{rf}_{i+\frac{1}{2}}$ is given as follows:

1) Choose $\alpha_{i+\frac{1}{2}} = \max_j |f'(u)|$, for all j in the stencil.
2) Get the point values $p^{i+\frac{1}{2}}_{j,+}$ and $p^{i+\frac{1}{2}}_{j,-}$ by Eq (10) for all j in the stencil, and set the cell averages $\overline{q}^{i+\frac{1}{2}}_{j,+} = p^{i+\frac{1}{2}}_{j,+}$ and $\overline{q}^{i+\frac{1}{2}}_{j,-} = p^{i+\frac{1}{2}}_{j,-}$, respectively.
3) Reconstruct the point values at the interface $r_{i+\frac{1}{2}}$, i.e., $q^l_{i+\frac{1}{2},+}$ and $q^r_{i+\frac{1}{2},-}$, by WENO reconstructions, based on the cell averages $p^{i+\frac{1}{2}}_{j,+}$ and $p^{i+\frac{1}{2}}_{j,-}$, respectively;
4) Get $\widehat{rf}_{i+\frac{1}{2}} = \alpha_{i+\frac{1}{2}}(q^l_{i+\frac{1}{2},+} - q^r_{i+\frac{1}{2},-})$.

Set $\lambda = \frac{\triangle t}{\triangle r}$, a finite difference scheme for Eq. (7) is

$$r_i u_i^{n+1} = r_i u_i^n - \lambda \left(\widehat{(rf)}_{i+\frac{1}{2}} - \widehat{(rf)}_{i-\frac{1}{2}} \right). \qquad (11)$$

The diffusion term can be discretized by central difference or other high order difference schemes. The whole scheme is direct forward to be extended to higher dimensions.

B. Temporal Discretization

After the space discretization, we get an ODE,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathrm{L}(u). \qquad (12)$$

The Total-Variation-Diminishing Runge-Kutta (TVDRK) proposed by Shu is used for time discretizaton [4].

III. Discretization of Poisson's Equation

The Poisson's equation is discretized by central finite difference scheme. The resulted linear equations can be solved by the fast FISHPACK.

IV. Simulation Results

We first test a double headed streamer discharge simulation in Nitrogen in a long parallel-plates applied with 52 kV voltage ($a = b = 1$ cm in Fig. (1)).

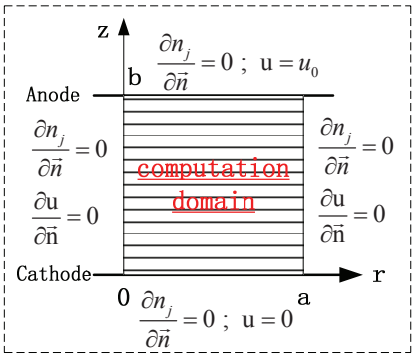
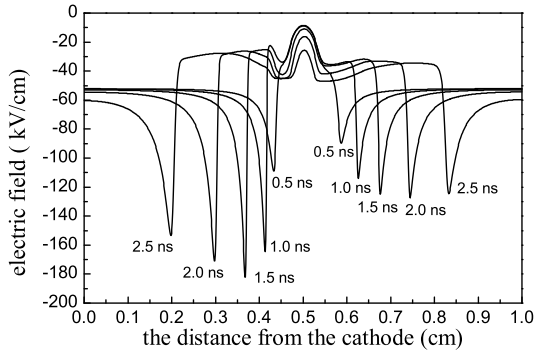
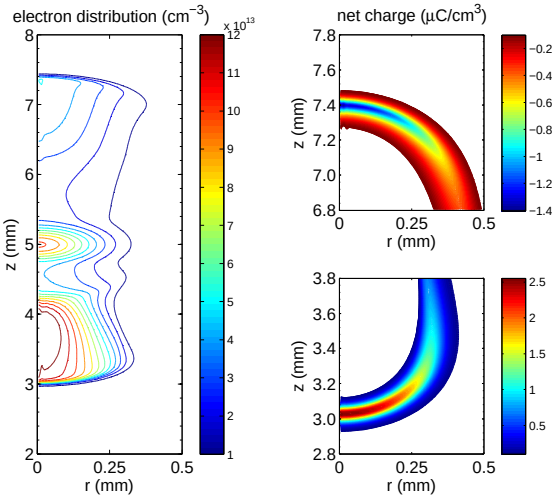


Fig. 1. the configuration of the discharge simulation

Fig 2(a) gives the electric field along the z-axis at different times(1 ns $= 10^{-9}$ s) and Fig (2(b)) shows charge densities distribution at $t = 2$ ns. The electric field is largely enhanced by the net charge and move towards the opposite electrode.



(a) electric field along z-axis



(b) charge densities distribution at $t = 2$ ns

Fig. 2. calculated electric field and charge densities distribution

V. Conclusion

This paper proposes a WENO finite difference method to simulate the fluid model of streamer discharges. A simulation of a double-headed streamer discharge in Nitrogen was performed using 2-dimensional fluid model. The results considering the exact photo-ionizations would be reported in the full version.

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