On the trajectory and rotation of a spherical magnet falling inside a conducting pipe

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***Abstract*—The trajectory of a spherical magnet falling inside a cylindrical, conducting pipe is modelled taking into account both the rotation of the magnet and its horizontal displacement. The electromagnetic force and torque are obtained via a *T-a* 3D FE formulation. The calculated trajectory and rotation are compared successfully to experimental data for different pipe diameters.**

***Index Terms*—Magnets, conducting materials, eddy currents, finite element methods.**

I. Introduction

It is well known that the free fall of a spherical magnet inside a cylindrical conducting pipe such as a copper tube is slowed down by electromagnetic forces. When only the vertical velocity component is considered and the magnetization direction of the magnet is assumed constant, the trajectory and velocity have been determined ([1] at steady-state and [2] taking into account the transient initial phase). These calculation have been refined by Donoso *et al* [3] who assumed that the magnetization direction is constant but allowed for a horizontal displacement. The rotation of the magnet in a pipe has been considered [4] when its motion is controlled.

In the real case, when the magnetization direction is not assumed constant, a spherical magnet will both rotate and have a non-zero horizontal displacement due to a lateral force. When the radius of the pipe is just slightly higher than that of the magnet, the magnetization direction has a damped oscillation around the axis of the pipe. However, when the radius of the pipe is much bigger than the radius of the magnet, the magnet rotates and the horizontal displacement is higher.

The purpose here is to construct a model and solve it so that the free fall of the magnet can be predicted in both the above cases. The general problem has 6 degrees of freedom. To decrease the model complexity, we assume that the rotation of the direction of the magnetization is possible only in a plane $(\vec{k}_x, \vec{k}_z)$. This simplified problem now has only 3 degrees of freedom (x_c, z, θ) (Fig. 1). The coupled differential equations form a dynamical system which is solved numerically and the results are compared to data obtained by experiments.

II. Governing equations

A moving reference frame with the z coordinate of the magnet center as the origin is chosen. The coordinates of the center of the spherical magnet are, in this frame, $\vec{x}_c = \vec{x} - (\vec{x}\cdot\vec{k}_z)\vec{k}_z$. Assuming that the motion is in the (k_x, k_z) plane, the velocity of the magnet $\vec{v}$, the magnetization direction $\vec{d}$, and rotating speed $\vec{\Omega}$ are:

$$\vec{v}(t) = \dot{x}(t)\,\vec{k}_x + \dot{z}(t)\,\vec{k}_z \tag{1}$$

$$\vec{d}(t) = \sin\theta(t)\,\vec{k}_x + \cos\theta(t)\,\vec{k}_z \tag{2}$$

$$\vec{\Omega}(t) = \vec{d}(t)\times\dot{\vec{d}}(t) = \dot{\theta}(t)\vec{k}_y \tag{3}$$

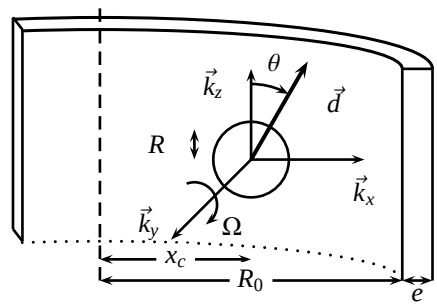

Figure 1: Magnet and pipe geometry.

The source magnetic vector potential (MVP) of the magnet in the region outside the magnet is:

$$\vec{a}_s(\vec{x},t) = \frac{\mu_0 M}{3}\frac{R^3}{|\vec{x}-\vec{x}_c(t)|^3}\left(\vec{d}(t)\times(\vec{x}-\vec{x}_c(t))\right). \tag{4}$$

If the time-derivative of the induced MVP $\vec{a}$ is neglected (due to the low speeds), then the equations are in this moving frame:

$$\vec{\nabla}\times\left(\frac{\vec{j}}{\sigma} - \dot{z}\vec{k}_z\times\left(\vec{\nabla}\times\vec{a}\right)\right) = \vec{\nabla}\times\left(-\partial_t\vec{a}_s + \vec{v}\times\vec{\nabla}\times\vec{a}_s\right) \tag{5}$$

$$\vec{a}(\vec{x},t) = \frac{\mu_0}{4\pi}\int_D \frac{\vec{j}(\vec{y},t)}{|\vec{x}-\vec{y}|}\,d\vec{y} \tag{6}$$

For convenience, the vector potential $\vec{T}$ of the current density is introduced $\vec{j} = \vec{\nabla}\times\vec{T}$. The system (5)-(6) is solved iteratively with a rapidly converging $\vec{T}-\vec{a}$ formulation. The Laplace force $\vec{f}$ and the corresponding torque $\vec{\Gamma}$ are then computed using:

$$\vec{f} = -\int_D \vec{\nabla}\times\vec{T}\times\vec{\nabla}\times\vec{a}_s\,d\vec{x} \tag{7}$$

$$\vec{\Gamma} = -\int_D (\vec{x}-\vec{x}_c)\times\vec{\nabla}\times\vec{T}\times\vec{\nabla}\times\vec{a}_s\,d\vec{x}. \tag{8}$$

The aim here is to develop a model with the force and torque as a function of the speeds $(\dot{x}, \dot{z}, \dot{\theta})$, in order to the study the off-axis, the lateral motion and the rotation of the magnet, and

their coupled effects. The $\vec{T}$ (and $\vec{d}$) fields are expanded into Taylor series as a function of $\dot{z}$:

$$\vec{T} = \sum_{n=0}^{\infty} \dot{z}^n (\vec{T}_{nx}\dot{x} + \vec{T}_{nz}\dot{z} + \vec{T}_{n\theta}\dot{\theta}). \tag{9}$$

The system (5)-(6) is transformed into simpler subsystems. The first-order fields $(\vec{T}_{0x}, \vec{T}_{0z}, \vec{T}_{0\theta})$ verify:

$$\Delta\vec{T}_{0x} = \sigma\partial_x\vec{\nabla}\times\vec{d}_s \quad \Delta\vec{T}_{0z} = \sigma\partial_z\vec{\nabla}\times\vec{d}_s \quad \Delta\vec{T}_{0\theta} = -\sigma\partial_y\vec{d}_s \tag{10}$$

and for higher order $\vec{T}_{ki}$ verify $\forall k \geq 1$:

$$\vec{d}_{(k-1)i} = \mathcal{F}(\vec{T}_{(k-1)i}) \quad \text{and} \quad \Delta\vec{T}_{ki} = \sigma\partial_z\vec{\nabla}\times\vec{d}_{(k-1)i} \tag{11}$$

where i stands for x, z or θ, and $\mathcal{F}$ corresponds to the Biot and Savart law applied to the $k-1$ order fields. The same Taylor series expansion as in (9) is used for force and torque. The coefficients are computed with a 3D FE method and some simplifications. A second order Taylor series expansion is sufficient because the vertical speed is sufficiently small.

A. On-axis analysis

The θ-dependence of the force and torque components analyzed for $x_c = 0$ are:

$$\begin{aligned} f_x &\simeq -A_x(1 - B_x\cos(2\theta))\dot{x} + C\sin(2\theta)\dot{z} - (H + I\cos(2\theta))\dot{\theta}\,\dot{z} \\ f_z &\simeq C\sin(2\theta)\dot{x} - A_z(1 + B_z\cos(2\theta))\dot{z} - D\sin(2\theta)\dot{\theta}\,\dot{z} \\ \Gamma_y &\simeq -(F - E\cos(2\theta))\dot{\theta} - (H + I\cos(2\theta))\dot{x}\,\dot{z} - D\sin(2\theta)\dot{z}^2 \end{aligned} \tag{12}$$

where (A_x, B_x, A_z, B_z) represent, respectively, the magnetic drag force for the lateral and vertical motion, C is the lateral force (source term of the lateral motion) and D is the main torque which drives the magnetization direction.

B. Off-axis analysis

The x_c-dependence is studied as a function of the fields with respect to the position. The source MVP (as well as $\vec{T}$) is expanded as a function of $\vec{x}_c = x_c\vec{k}_x$:

$$\vec{d}_s(\vec{x}, \vec{x}_c) = \vec{d}_s(\vec{x}, \vec{0}) + \vec{\vec{\nabla}}\vec{d}_s(\vec{x}, \vec{0}) \cdot \vec{k}_x\ x_c \tag{13}$$

$$\vec{T} = \vec{T}(x_c = 0) + x_c\vec{T}'(x_c = 0) \tag{14}$$

The first order term $\vec{T}'(x_c = 0)$ is used to analyze the x_c-dependence, and is expanded as (9). Then the θ-dependence of the first-order terms of $(\vec{f}', \vec{\Gamma}')$ is:

$$\begin{aligned} f'_x &\simeq -G\sin(2\theta)\,\dot{\theta} - K\,\dot{z}^2 \qquad f'_z \simeq -L\,\dot{\theta} - K\,\dot{x}\,\dot{z} \\ \Gamma'_y &\simeq -G\sin(2\theta)\,\dot{x} - L\,\dot{z} \end{aligned} \tag{15}$$

where (G, K) represent, for the lateral motion, the restoring forces due to $(\dot{\theta}, \dot{z})$, and L is a friction torque coefficient.

The ordinary differential equations to be solved are:

$$\frac{d}{dt}\begin{pmatrix} x_c \\ m\dot{x} \\ z \\ m\dot{z} \\ \theta \\ J\dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ f_x + x_c f'_x \\ \dot{z} \\ mg + f_z + x_c f'_z \\ \dot{\theta} \\ \Gamma_y + x_c\Gamma'_y \end{pmatrix} \tag{16}$$

where m is the magnet mass, and J is its moment of inertia.

III. Results

The simplified model (16) is solved for the case of a spherical Nd-Fe-B magnet ($R = 6.35mm$, $m = 8g$, $\mu_0 M = 0.7T$) and copper tubes ($\sigma = 5\ 10^7(\Omega.m)^{-1}$) of different radii ($8mm \leq R_0 \leq 16mm$). When the tube radius is $R_0 = 8mm$ (Fig. 2 left), the magnet position slowly oscillates around the center of the pipe. This is also the case for the magnetization direction. The period of the oscillations of θ is about $2s$, this is due to the lateral displacement of the magnet. If the lateral displacement was neglected, then the period would be around ten times higher. These values have been compared to experimental data. The measured vertical speed is about $0.07m/s$, and the period of oscillation of θ is about $2s$, in total agreement with our calculations.

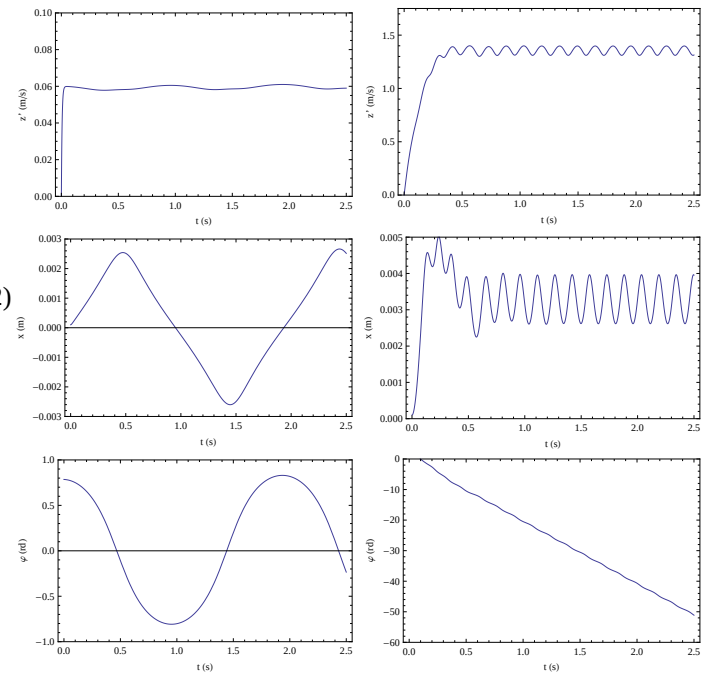

Figure 2: $(\dot{z}, x_c, \theta)$ (from top to bottom), for a pipe radius of $R_0 = 8mm$ (left) and $R_0 = 16mm$ (right) versus time

When the copper tube radius is $R_0 = 16mm$ (Fig. 2 right), the spherical magnet oscillates around a mean vertical off axis position, which is not zero and rotates at an almost constant speed. Experiments confirm the magnet rotation and the order of magnitude of the calculated speed (slightly more than 1m/s, which much higher than for the pipe of radius $R_0 = 8mm$).

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