A posteriori error estimation in stochastic static problems

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***Abstract*— To solve stochastic magnetostatic problem, magnetic fields can be approximated using a double discretization: the spatial discretization using the finite element method for example and the discretization of the stochastic dimension using a polynomial chaos expansion for example. In this paper, we aim at determining the numerical error due to the stochastic discretization.**

***Index Terms*—Stochastic processes, estimation error, magnetostatics, finite element method.**

I. INTRODUCTION

To quantify the impact of uncertainties of the input data on the output data of a model, stochastic approaches can be used which consists in modeling the uncertain inputs as random variables. The outputs of the model are then also random variables or fields. If the model is based on a Partial Differential Equations (PDE) system like the Maxwell equations in electromagnetism, then a Stochastic PDE system has to be solved. The discretization of the spatial dimension using the Finite Element Method (FEM) for example yields a stochastic matrix system to solve. The degrees of freedom, potential values at the nodes, are not scalars anymore but random variables. Several methods are available in the literature to characterize these random variables. The Monte Carlo Simulation method (MCSM) is probably the best known and is widely used in different scientific areas (financial mathematics, biostatistics, mechanics, etc.). The MCSM is robust and simple but very time consuming especially when coupled with a numerical model such as a Finite Element model. Approximation methods which consist in searching the random degrees of freedom in a finite dimension functional space have been introduced in the early 90's. The Polynomial Chaos Expansion (PCE) is one of the most popular approximation method [1]. These double discretization yields an approximated solution whose accuracy depends on the discretization along the spatial dimension (mesh) and the discretization along the random dimension (the order of the PCE). To evaluate the numerical error (i.e. the distance between the numerical solution and the exact solution), one can distinguish two kinds of error estimation: *a priori* error and *a posteriori* error estimation. In this paper, we are interested in the *a posteriori* error estimation which is function of the numerical solution and so, the error estimation is evaluated after the numerical resolution of the problem. The *a posteriori error* estimation has been already addressed in literature [2, 3]. In [2], a global error estimation based on the hyper-circle theorem is proposed. This error estimation requires the solutions of two complementary formulations. This error estimator takes into account simultaneously the error due to the space discretization and along the random dimensions. In [3], an error estimator enabling to evaluate the error due the stochastic discretization (stochastic error) has been proposed based on an enrichment of the functional space.

In this paper, we propose a stochastic error estimator based on the evaluation of the residue calculated from the obtained numerical solution. The proposed error estimation is tested on an academic example.

II. MAGNETOSTATIC PROBLEM WITH UNCERTAINTIES ON THE BEHAVIOR LAW

We are interested in a magnetostatic problem defined in a domain D with uncertainties on the permeability. The uncertain permeability can be modeled by a random field $\mu(x,\theta)$ where the parameter θ refers to a random outcome. We assume that the random fields $\mu(x,\theta)$ can be expressed explicitly as a function of a random vector $\xi(\theta) = (\xi_1(\theta), \xi_2(\theta),..., \xi_M(\theta))$ defined on $\Theta^M \subset \mathbb{R}^M$ where $\xi_1(\theta), \xi_2(\theta),..., \xi_M(\theta)$ are real independent random variables with known probability density functions. In the following, to simplify the notations, the dependency of the random vector ξ on θ will be removed. Then, the permeability take the form $\mu(x,\xi)$. We assume also that the permeability is bounded:

$$0 < \mu_{\min}(x) \le \mu(x,\xi) \le \mu_{\max}(x) < \infty \quad \forall \xi \in \Theta^M . \qquad (1)$$

Consequently, the magnetic field $\mathbf{H}$ and the magnetic flux density $\mathbf{B}$ are also function of ξ. The stochastic magnetostatic problem on a domain D can be written:

$$\begin{cases} div\,\mathbf{B}(x,\xi) = 0 \\ \mathbf{curl}\,\mathbf{H}(x,\xi) = \mathbf{J}_s(x) \\ \mathbf{B}(x,\xi) = \mu(x,\xi)\cdot\mathbf{H}(x,\xi) \end{cases} \qquad (2)$$

where $\mathbf{J}_s$ is the source term that can be written as $\mathbf{J}_s(x)=\mathbf{curl}\mathbf{H}_s(x)$. For completeness, boundary conditions are also added. The problem (2) can be solved using the scalar potential formulation where the scalar potential $\Omega(x,\xi)$ is defined such that $\mathbf{H}(x,\xi)=-\mathbf{grad}\Omega(x,\xi)+\mathbf{H}_s(x)$. The domain D is spanned by a tetrahedral mesh $\mathcal{M}$ with n_0 nodes, n_1 edges, n_2 facets and n_3 elements. Let denote $\Omega^{h0}(x,\xi) \in V_x^{h0}$ the approximation of $\Omega(x,\xi)$ which satisfies:

$$\begin{aligned} &\int_D \mathbf{grad}\Omega^{h0}(x,\xi)\cdot\mu(x,\xi)\cdot\mathbf{grad}w_{0i}(x)dx = \\ &\int_D \mathbf{H}_s(x)\cdot\mu(x,\xi)\cdot\mathbf{grad}w_{0i}(x)dx \ \text{ for } i=1:n_0, \forall\xi\in\Theta^M \end{aligned} \qquad (3)$$

where $w_{0i}(x)$ is the shape function associated to the node i [4] and V_x^{h0} is defined by:

$$V_x^{h0} = \text{span}\{w_{0i}(x) / i = 1,2,...,i_0-1,i_0+1,...n_0\} \qquad (4)$$

with i_0 a given node of the mesh $\mathcal{M}$. Generally, $\Omega^{h0}(x,\xi)$ is not available because (3) should be solved for an infinite number of values of $\xi \in \Theta^M$. In the following section, a method to approximate $\Omega^{h0}(x,\xi)$ by using a PCE is briefly presented.

III. APPROXIMATION USING A PCE

The scalar potential approximation $\Omega^{h0,P}(x,\xi)$ is sought in the following form:

$$\Omega^{h0,P}(x,\xi) = \sum_{i=1,i\ne i_0}^{n_0}\sum_{\alpha=0}^{P}\Omega_{i\alpha}\Psi_\alpha(\xi)w_{0i}(x) \qquad (5)$$

where $\{\Psi_\alpha(\xi)\}_{0\le\alpha\le P}$ is a set of multivariate polynomials of order at most p (truncated PCE) [1] and $\Omega_{i\alpha}$ the coefficients to determine. Here, the order of a polynomial chaos is equal to the sum of the orders of the monodimensional polynomials. In the SSFEM method [5], the coefficients $\mathbf{\Omega}_{i\alpha}$ are determined by:

$$\mathbb{E}\left(\int_D \mathbf{grad}(\Omega^{h0,P}(x,\xi))\mu(x,\xi)\mathbf{grad}(w_{0i}(x))dx \cdot \Psi_\alpha(\xi)\right) = \mathbb{E}\left(\int_D \mathbf{H}_s(x)\mu(x,\xi)\mathbf{grad}(w_{0i}(x))dx \cdot \Psi_\alpha(\xi)\right) \quad (6)$$

with $\mathbb{E}()$ the expectation. The equation (6) leads to a linear matrix system of dimension $(n_0-1)\times(P+1)$ where the solution is the vector of the coefficients $\mathbf{\Omega}_{i\alpha}$. In our case, an iterative solver will be used to solve the linear system with a stopping criterion ε on the residue of the linear system (6). The solution obtained with the SSFEM depends on the spatial mesh, on the order p of the PCE set and also on the stopping criterion ε. Therefore, the numerical error depends also on these three factors. In the following section, for a given mesh, a stochastic error estimator (partial error due to the discretization using the truncated PCE) which evaluates the "distance" between $\mathbf{\Omega}^{h0,P}(x,\xi)$ and $\mathbf{\Omega}^{h0}(x,\xi)$ is proposed.

IV. *A POSTERIORI* STOCHASTIC ERROR ESTIMATION

The stochastic error is defined by:

$$e_{sto}^2(\xi) = \int_D \left| \mu(x,\xi)^{\frac{1}{2}} \mathbf{grad}(\Omega^{h0,P}(x,\xi) - \Omega^{h0}(x,\xi)) \right|^2 dx . \quad (7)$$

We propose the following error estimation:

$$k_1 \boldsymbol{r}^t(\xi)\cdot\Lambda_0^{-1}\cdot\boldsymbol{r}(\xi) \le e_{sto}^2(\xi) \le k_2 \boldsymbol{r}^t(\xi)\cdot\Lambda_0^{-1}\cdot\boldsymbol{r}(\xi) \quad (8)$$

where $\boldsymbol{r}(\xi)$ is the residual vector and Λ_0 is the mean value of the stiffness matrix that are defined by:

$$\begin{aligned}
&\Lambda_0 = \left[\Lambda_{0ij}\right],\ i,j = 1:n_0 \setminus i_0 \quad \text{with} \\
&\Lambda_{0ij} = \int_D \mathbf{grad} w_{0i}(x)\mathbb{E}\left(\mu(x,\xi)\right)\mathbf{grad} w_{0j}(x)dx \\
&\boldsymbol{r}(\xi) = \left[r_i(\xi)\right],\ i = 1:n_0 \setminus i_0 \quad \text{with} \\
&r_i(\xi) = \int_D \mathbf{grad}\Omega^{h0,P}(x,\xi)\mu(x,\xi)\mathbf{grad} w_{0i}(x)dx \\
&\qquad - \int_D \mathbf{H}_s(x)\cdot\mu(x,\xi)\cdot\mathbf{grad}(w_{0i}(x))dx
\end{aligned} \quad (9)$$

and k_1, k_2 the coefficients that depend only on $\mu(x,\xi)$:

$$k_1 = \min_{x\in D}\left(\frac{\mathbb{E}\left[\mu(x,\xi)\right]}{\mu_{\max}(x)}\right); k_2 = \max_{x\in D}\left(\frac{\mathbb{E}\left[\mu(x,\xi)\right]}{\mu_{\min}(x)}\right). \quad (10)$$

One can notice that the coefficients k_1 and k_2 can be calculated explicitly. In practice, the ratio between the upper bound and the lower bound of the stochastic error (8) is about several units. Furthermore, the estimation (8) is independent on the method used to solve the stochastic problem (SSFEM, Non intrusive, etc.) and on the choice of the stochastic approximation basis (truncated PCE in this paper). For example, even with an approximation based on wavelets [6] instead of a truncated PCE, the estimation (8) holds.

V. NUMERICAL EXAMPLE

The magnetostatic example is presented in Fig. 1. The domain is split up into 5 sub-domains with the relative permeabilities μ_0=1, μ_1=μ_2=1000. μ_3 and μ_4 are two independent uniform random variables defined on [600-1400]. The current $\mathbf{J}_s$ is imposed equal 1A. We use the SSFEM method to solve this problem (see (6)). With a given $\Omega^{h0,P}(x,\xi)$, the mean value of stochastic error estimation is compared to the mean value of the stochastic error obtained by the Monte-Carlo method (MCSM). With the MCSM, the stochastic error at each point $\boldsymbol{\xi_k}$ is evaluated by calculating the distance between $\Omega^{h0,P}(x,\boldsymbol{\xi_k})$ and $\Omega^{h0}(x,\boldsymbol{\xi_k})$ obtained by solving (3) at the value $\boldsymbol{\xi} = \boldsymbol{\xi_k}$.

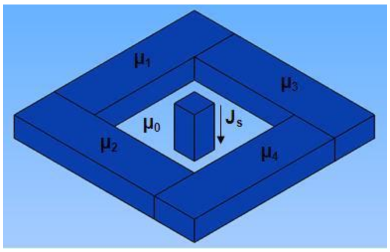

Fig. 1. Stochastic magnetostatic problem

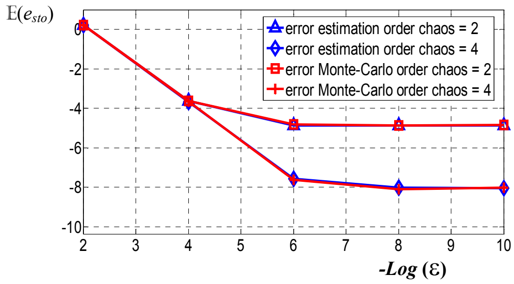

Fig. 2. Expectation of the estimator and of the error approximated by the Monte Carlo method in function of the stopping criteria ε

In the Fig. 2 we represent the evolution of the stochastic error obtained by the proposed estimator and the one obtained by the MCSM in function of the order of the PCE and of the stopping criterion with a fixed mesh of 2617 nodes. We can notice that the error obtained by the proposed method and the error obtained by the MCSM are very close and these errors decrease when reducing the stopping criterion ε. We can notice also that the stochastic error is almost constant from a given value of the stopping criterion (log(ε) = -6 with order p = 2 and log(ε) = -8 with order p = 4). It is not necessary to increase the precision of the resolution of the linear system (6) beyond this point by reducing ε.

VI. REFERENCES

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