Non-Asymptotic Homogenization of Electromagnetic Metamaterials via Discrete Hodge Operators with Trefftz Calibration

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***Abstract*—The paper establishes a connection between the recently proposed homogenization methodology for electromagnetic metamaterials with the theory of discrete Hodge operators. The methodology also makes extensive use of Trefftz bases (functions that satisfy the underlying differential equation locally and tend to approximate the solution well). All these ideas have been extensively explored in other areas but are now applied to homogenization. Discrete Hodge operators are linear mappings between the pairs of suitably defined coarse-grained fields viewed as differential forms. Several definitions of discrete Hodge operators are considered, with a focus on preserving the energy or, more generally, the bilinear form associated with the underlying differential equation. The theory is *non-asymptotic*, i.e. applicable to arbitrary lattice cell sizes (not necessarily vanishingly small), which is essential in the analysis of nontrivial effects ("artificial magnetism," negative refraction, cloaking).**

***Index Terms*—Homogenization, discrete Hodge operators, metamaterials, effective material parameters, Trefftz functions**

I. Introduction

This paper fuses together three ideas that have been the subject of much attention in the Compumag community over the years and applies these ideas in a new area: homogenization of electromagnetic metamaterials. The first idea, the use of *discrete differential forms* and the closely related *div- and curl-conforming interpolation* of fields, has been extensively explored since the early 1980s in the context of edge and face elements [1]–[3]. It was later understood that many computational methods can be obtained by combining the exact discrete representation of Ampere's and Faraday's laws with suitable *discrete Hodge operators* – linear mappings between discretized fields (viewed as differential forms) [4]–[5]. The third idea is *Trefftz approximations*. By definition, Trefftz functions satisfy the underlying differential equation with the relevant boundary conditions and typically yield much higher approximation accuracy of the solution than traditional piecewise-polynomial bases. This has been successfully exploited in a variety of techniques including pseudospectral methods, generalized FE and FD algorithms (e.g. [6]–[9]).

Homogenization (theory of effective material parameters) of metamaterials indeed constitutes a new application area for all these time-tested ideas. Metamaterials are periodic dielectric/metal structures whose lattice cell size is smaller than the vacuum wavelength but not vanishingly small, leading to unusual resonant characteristics and intriguing effects (see e.g. review [10]). Traditional homogenization methods that work well when the lattice cell size a tends to zero may be impossible to apply when the cell size is appreciable, which is a necessary condition for the nontrivial physical behavior [11]. This motivates the *non-asymptotic* homogenization method of [12]–[14]. The purpose of the present paper is to establish a connection of this method with the theory of discrete Hodge operators [4] and to compare different options for defining such an operator.

II. Discrete Energy Hodge Operators with Trefftz Calibration

We describe metamaterials, in the frequency domain, as periodic structures with given intrinsic electromagnetic parameters $\epsilon(\omega, \mathbf{r})$, $\mu(\omega, \mathbf{r})$ within each lattice cell. The size of a metamaterial sample is sufficiently large but finite. In the absence of sources, Maxwell's equations for the spatially-rapid fields $\mathbf{e}, \mathbf{h}, \mathbf{d}, \mathbf{b}$ are $\nabla \times \mathbf{e} = -j\omega\mathbf{b}$; $\nabla \times \mathbf{h} = j\omega\mathbf{d}$, with $\mathbf{d} = \epsilon\mathbf{e}$, $\mathbf{b} = \mu\mathbf{h}$. The objective is to put forward a suitable approximation of the rapid fields by coarse-grained ones, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$, and then to find the corresponding material tensor, i.e. a linear map relating $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$. The critical constraint is that Maxwell's equations must hold on the coarse level, $\nabla \times \mathbf{E} = -j\omega\mathbf{B}$; $\nabla \times \mathbf{H} = j\omega\mathbf{D}$, along with a linear material relation of the form $\mathcal{L} : (\mathbf{E}, \mathbf{H}) \to (\mathbf{D}, \mathbf{B})$. One may seek $\mathcal{L}$ as

$$\mathbf{D} = \epsilon_{\text{eff}}\mathbf{E} + \xi\mathbf{H}; \quad \mathbf{B} = \mu_{\text{eff}}\mathbf{H} + \zeta\mathbf{E}$$

where ϵ_{eff}, ξ, ζ, μ_{eff} are in general tensorial. This relationship includes magnetoelectric coupling parameters ξ, ζ commonly used in metamaterial science; note, however, that the linear map $\mathcal{L}$ can be even more general and may represent a nonlocal relationship [14], when fields at a given point depend on fields in its neighborhood. The reciprocity ($\xi = \zeta^*$) may or may not be explicitly built into the model *a priori*.

The fact that the coarse-grained fields must satisfy Maxwell's equations dictates that $\mathbf{E}, \mathbf{H} \in H_{\text{curl}}(\Omega)$; $\mathbf{D}, \mathbf{B} \in$

$H_{\mathrm{div}}(\Omega)$. where H_{curl} and H_{div} are standard spaces of square-integrable complex functions with a square-integrable curl or div, respectively, in a given computational domain Ω.

There are three key ingredients in the overall homogenization procedure: (i) The approximation space for fields. We focus on Trefftz bases due to their excellent approximation properties, as evidenced by extensive experience [8]. (ii) Div- and curl-conforming interpolations $\mathcal{I}_{\mathrm{curl}}$ and $\mathcal{I}_{\mathrm{div}}$ from the field circulations and fluxes on the boundary of a lattice cell into its volume. Whitney-like interpolation is a natural choice, lattice cells being analogous to edge/face "elements". Alternatively, div- and curl-conforming *Trefftz* interpolations can also be considered. (iii) The linear map $\mathcal{L}$ that, in the language of differential forms, is equivalent to a discrete Hodge operator [4].

Depending on one's choices in the items above, one obtains a family of homogenization procedures similar in principle but different with respect to complexity and accuracy. One possibility is a "Galerkin Hodge" based on constant material properties, whereby the bilinear forms corresponding to the problems on the fine and coarse levels are required to be equal for all test ("calibration") fields in a suitable Trefftz subspace $\mathcal{T}(C)$ of all fields in a cell C:

$$\mathcal{F}^{EH}(\psi^{EH}, \tilde{\psi}^{EH}) \overset{l.s.}{=} \mathcal{F}^{eh}(\psi^{eh}, \tilde{\psi}^{eh}), \quad \tilde{\psi}, \tilde{\psi}^{eh} \in \mathcal{T}(C) \qquad (1)$$

with $\psi^E = \mathcal{I}_{\mathrm{curl}}\psi^e$; $\psi^H = \mathcal{I}_{\mathrm{curl}}\psi^h$, where ψ^{EH} is a vector combining the coarse fields $\mathbf{E}$, $\mathbf{H}$, and similarly for ψ^{eh}; the tilde sign denotes test functions. 'l.s.' in (1) indicates a least squares solution in a fixed trial space. The Trefftz space $\mathcal{T}(C)$ may be spanned e.g. by a set of Bloch waves traveling in different directions. The bilinear forms above are defined as

$$\mathcal{F}(\psi^{EH}, \tilde{\psi}^{EH}) = (\epsilon_{\mathrm{eff}}\psi^E, \tilde{\psi}^E) + (\zeta\psi^H, \tilde{\psi}^E) + (\xi\psi^E, \tilde{\psi}^H) + (\mu_{\mathrm{eff}}\psi^H, \tilde{\psi}^H)$$

$$\mathcal{F}^{eh}(\psi^{eh}, \tilde{\psi}^{eh}) = (\epsilon\psi^e, \tilde{\psi}^e) + (\mu\psi^h, \tilde{\psi}^h)$$

Here $(\cdot, \cdot)$ is the L_2 inner product in the lattice cell.

III. A Numerical Example

As an illustrative example, we consider wave propagation through a layered slab; an exact analytical solution is available in this case for comparison and error estimation. The lattice cell contains two nonmagnetic layers with the widths $w_1 = 0.75$, $w_2 = 0.25$ (the cell size $a = 1$) and permittivities $\epsilon_1 = 1$, $\epsilon_2 = 4$, respectively. The speed of light is normalized to unity. The thickness of the slab is $d = 20$, and incidence is normal. The Trefftz calibration space $\mathcal{T}(C)$ in this case is spanned by two independent waves that are easy to find analytically. The interpolation space is also chosen to be spanned by Trefftz functions, but those that correspond to a homogeneous medium inside the cell. This can be shown to produce not only an "ideal Galerkin" scheme with a zero consistency error but also "ideal" effective parameters of the slab for which transmission/reflection are modeled *exactly* (error at the roundoff level in Fig. 1). In contrast, the quasi-static material tensor $\epsilon_{\mathrm{static}} = w_1\epsilon_1 + w_2\epsilon_2$, $\mu_{\mathrm{static}} = 1$ leads to significant errors and ultimately to meaningless results as the

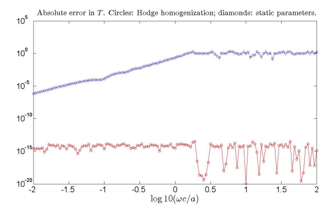

Figure 1: Error in the transmission coefficient: roundoff level for proposed homogenization (red) vs. static material tensor (blue).

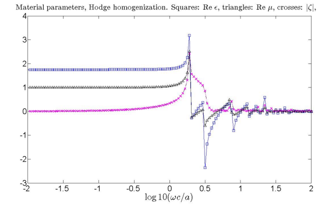

Figure 2: Material parameters obtained by Trefftz calibration. Blue: Re ϵ_{eff}, black: Re μ_{eff}, magenta: $|\zeta| = |\xi|$.

frequency increases (Fig. 1). Material parameters obtained by Trefftz homogenization are shown in Fig. 2.

IV. Conclusion

A new homogenization framework for electromagnetic metamaterials includes Trefftz approximations of the fields, div- and curl-conforming interpolations, and the respective discrete Hodge operators. Several options for the Hodge are considered and will be further discussed in an extended paper. Unlike traditional homogenization techniques, the proposed approach is non-asymptotic, i.e. applicable to arbitrary lattice cell sizes, not necessarily vanishingly small.

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