The Face-Based Gradient Smoothing Point Interpolation Method applied to 3D Electromagnetics

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***Abstract*—This work presents the Face-Based Gradient Smoothing Point Interpolation Method (FS-PIM), a numerical method derived from the Point Interpolation Method that solves three-dimensional boundary value problems. FS-PIM is supported by the theory of G-space, weakened-weak formulations and the gradient smoothing operation. The obtained results show that both convergence rate and accuracy of the approximation generated by FS-PIM are better than the ones presented by the Finite Element Method.**

***Index Terms*—Meshless methods, point interpolation methods, gradient smoothing, weakened-weak form, three-dimensional.**

I. Introduction

The development of the Point Interpolation Method (PIM) and its application in electromagnetics have increased in the last years. Efforts have been made on frequency and time-domain modeling with methods based on strong forms [1], [2], on formulations based on weak forms combined with global Galerkin methods [3] and on local Petrov-Galerkin methods [4]. Parallel algorithms on graphics processing units (GPUs) has also been proposed to reduce the processing time [5].

A new PIM approach that works with weakened-weak forms (W^2) was proposed by [6]. This class of methods uses the operation of gradient smoothing to approximate the derivatives of the field function in global Galerkin weak forms [7]. With such modifications, a weakened-weak formulation arises overcoming the problem of the incompatibility of PIM shape functions that impacts negatively the quality of the numerical solution [3]. The W^2 form also brings to PIM convergence rates higher than the Finite Element Method (FEM) with more accurate approximations [6].

PIM with W^2 formulation is used in [8] for solving two-dimensional electromagnetic problems. In that work, the gradient smoothing operation is performed based on nodes, edges and cells. The results proved the capacity of those methods to generate very accurate solutions, especially the edge based method. However, a three-dimensional electromagnetic application does not exist yet. This work presents a 3D Point Interpolation Method based on gradient smoothing with weakened-weak formulation. The method is applied to an electrostatic problem for validation purposes. Finally, the results are compared to those obtained by the FEM.

II. Mathematical Formulation

A. Gradient Smoothing and Weakened-Weak Form

Three-dimensional electrostatic problems are formulated as follows: given a permittivity ϵ and a volume charge density ρ, determine the scalar potential $V : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ that satisfies

$$\begin{aligned} \nabla \cdot (\epsilon \nabla V) &= \rho && \text{on} \quad \Omega \\ V &= g && \text{on} \quad \Gamma_g \\ -\epsilon \frac{\partial V}{\partial n} &= h && \text{on} \quad \Gamma_h \end{aligned} \tag{1}$$

where g and h are the Dirichlet and Neumann boundary conditions, respectively, on the boundaries Γ_g and Γ_h, and $\partial\Omega = \Gamma = \Gamma_g \cup \Gamma_h$ with $\Gamma_g \cap \Gamma_h = \emptyset$.

The weak form of the problem (1) can be obtained using the weighted residual method and it is expressed as:

$$\int_\Omega \epsilon \nabla w \cdot \nabla V \; d\Omega = -\int_\Omega w\rho \; d\Omega - \int_{\Gamma_h} wh \; d\Gamma, \quad \forall w \in U \tag{2}$$

where w is the test function of the weighted residual method and U is the test function space, in which the function must belong to $\mathbb{H}^1(\Omega)$, i.e., its first order derivatives must be square integrable over Ω.

The gradient of V can be approximated by an integral representation in a predefined domain called smoothing domain Ω_x^s [6]

$$\hat{\nabla} V = \int_{\Omega_x^s} \nabla V \hat{W} d\Omega = \int_{\Omega_x^s} \nabla(V\hat{W}) d\Omega - \int_{\Omega_x^s} V \; \nabla \hat{W} d\Omega \tag{3}$$

where $\hat{W}$ is called smoothing function. Applying the gradient theorem in (3), we have

$$\hat{\nabla} V = \int_{\Gamma_x^s} V\hat{W} \; \vec{n} \, d\Omega - \int_{\Omega_x^s} V \; \nabla \hat{W} d\Omega \tag{4}$$

where $\Gamma_x^s = \partial\Omega_x^s$ and $\vec{n}$ is the unit outwards normal on Γ_x^s.

Considering the smoothing function $\hat{W}$ locally constant over Ω_x^s and equals to the inverse of its volume V_x^s, then (4) is simplified to

$$\hat{\nabla} V = \frac{1}{V_x^s} \int_{\Gamma_x^s} V \; \vec{n} \, d\Gamma \tag{5}$$

which is the smoothed gradient of V in Ω_x^s.

The next step is to divide the problem domain into N_s smoothing domains with no overlap. Therewith, we can approximate the gradient in the weak form (2) by the smoothed gradient (5), resulting in the weakened-weak form

$$\sum_{i=1}^{N_s} \epsilon V_i^s \ \hat{\nabla} w \cdot \hat{\nabla} V = -\int_\Omega w\rho \ d\Omega - \int_{\Gamma_h} wh \ d\Gamma, \quad \forall w \in U_G \quad (6)$$

where V_i^s is the volume of the ith smoothing domain and U_G is the test function space, which now must be in $\mathbb{G}^1(\Omega)$ [7]. This means that only the function (and not its derivatives) must be square integrable over Ω, a weaker requirement with respect to the test functions of the weak form. The integration of the weakened-weak form is carried out using the smoothing domains already constructed.

B. Point Interpolation Method

The scalar potential V in (6) is approximated by the Point Interpolation Method [6]. PIM uses linear polynomials in the basis to generate the shape functions with consistency C^1. The approximation $V^h(\mathbf{x})$ at point $\mathbf{x}$ is given by

$$V^h(\mathbf{x}) = \sum_{i=1}^{n} p_i(\mathbf{x}) a_i \quad (7)$$

where a_i is the coefficient for the ith polynomial term p_i and n is the number of nodes in the support domain of $\mathbf{x}$.

The support domain is a set of nodes in the vicinity of point $\mathbf{x}$ used to compute the shape functions. In this work, T4 scheme is used for support nodes selection [6]. T4 scheme uses the tetrahedral integration mesh and selects the 4 vertices of the cell where the point is located as the support nodes.

C. Construction of the Smoothing Domains

To compute the smoothed gradients, it is necessary to build the smoothing domains. As seen in Section II-A, overlaps are forbidden and the union of all smoothing domais must cover the entire problem domain.

The smoothing domains are constructed based on the triangular faces of the tetrahedral mesh. For each face f, a smoothing domain Ω_f^s is created connecting the vertices of f to the centroids of the cells adjacent to f.

III. Numerical Results and Conclusions

In this section we test FS-PIM against an electrostatic problem with a unit cubic domain having a known analytical solution. The top face is at an electric potential that varies sinusoidally from 0V to 10V. The other faces are at a potential of 0V. No electric charge is present in the problem.

The solution is computed using 5 different tetrahedral meshes with equally spaced nodes. The error is computed for the potential V at arbitrally sampled points. We also test FEM with the same meshes for comparison. Figure 1 shows the errors for both methods.

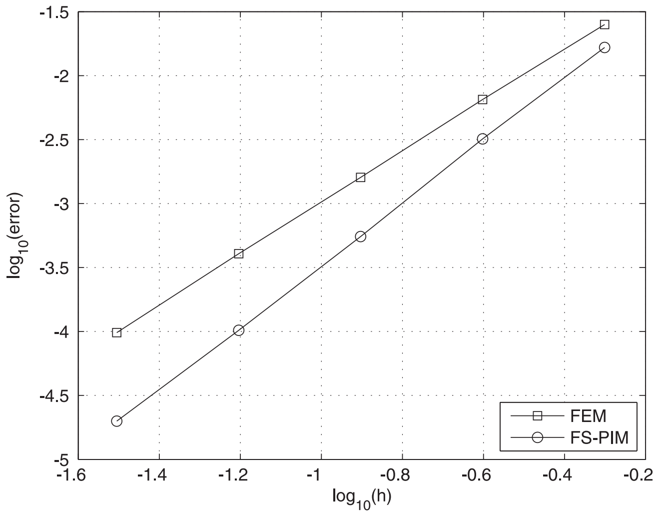

Figure 1: Numerical solution error for FS-PIM and FEM. h is the distance between nodes.

It can be seen that FS-PIM generates better approximations than FEM. While the latter presents a convergence rate of 2.0, as expected for linear models based on weak forms, the former presents a convergence rate of 2.4, confirming the presence of superconvergence for models based on weakened-weak forms [8], [7].

The treated problem indicates FS-PIM as a good alternative to FEM in 3D electromagnetics. More realistic problems that show the robustness of the method even in the presence of low quality mesh elements will be presented in the extended version of this paper.

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