A Path Toward Stable Higher Order Discretization of Constitutive Equations in FIT

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***Abstract*—This article details a novel energetic approach for achieving stable higher order discretizations of constitutive equations in the Finite Integration Technique (FIT) and in its cousin, the Cell Method (CM), in the basic case of two-dimensional discretization of frequency domain electromagnetic problems over pairs of non-uniform Cartesian dual grids.**

***Index Terms*—Computational electromagnetics, Finite Integration Technique, Cell Methods, Higher Order Discretization.**

I. Introduction

FIT and CM are discretization methods variously applied in computational electromagnetics. As it is well known, these methods are based on the discretization of space by means of a pair of *dual* grids. Integral electromagnetic quantities are associated to proper oriented geometric elements of the dual grids in such a way that electromagnetic balance equations can be naturally discretized as exact equations. The discretization of constitutive equations is the key point as it is the sole responsible of the accuracy and stability properties of the numerical method [1]. Various efforts are reported in literature for discretizing constitutive equations over different classes of dual grids. A particularly convenient approach is the *energetic* method introduced in [2], since it theoretically ensures consistency, stability and convergence of the discretizations over arbitrary polyhedral dual grids [3], [4].

Despite many recent results reported in literature for discretizing constitutive equations, virtually no clue [5] exists on how to increase the order of discretization with respect to the first order. In this paper this problem is attacked by a novel approach which extends the energetic method [2]. The details of such approach are provided in the basic case of the second-order discretization of constitutive equations over pairs of non-uniform Cartesian dual grids for two-dimensional frequency domain electromagnetic problems.

II. Second Order Discretization of Constitutive Equations

A two-dimensional electromagnetic problem in the frequency domain is analyzed within the spacial region Ω. Without loss of generality, let the electric field $\mathbf{E}(\mathbf{r})$ and the electric displacement $\mathbf{D}(\mathbf{r})$, functions of the position vector $\mathbf{r} = (x, y)$, be tangent to Ω, and let the magnetic field $\mathbf{H}(\mathbf{r})$ and the magnetic induction $\mathbf{B}(\mathbf{r})$ be normal to Ω.

The Ω region is discretized by a Cartesian primal grid $\mathcal{G}$ and a Cartesian dual grid $\tilde{\mathcal{G}}$ and degrees of freedom (dofs) are introduced as follows. Let $\boldsymbol{e}$ be the array of the circulations e_i of $\mathbf{E}(\mathbf{r})$ along the n_e edges of $\mathcal{G}$ and let $\boldsymbol{b}$ be the array of

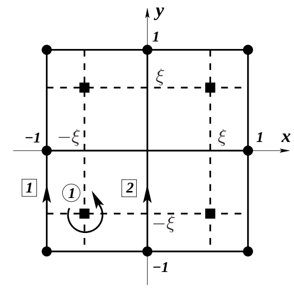

Figure 1: Pair of dual grids $\mathcal{G}$, $\tilde{\mathcal{G}}$. Oriented primal edges 1, 2 and oriented primal face 1 are outlined.

the fluxes b_i of $\mathbf{B}(\mathbf{r})$ across the n_f faces of $\mathcal{G}$. Also let $\tilde{\boldsymbol{d}}$ be the array of the fluxes $\tilde{d}_i$ of $\mathbf{D}(\mathbf{r})$ across the $n_{\tilde{e}} = n_e$ edges of $\tilde{\mathcal{G}}$ and let $\tilde{\boldsymbol{h}}$ be the array $\tilde{h}_i$ of the normal component of $\mathbf{H}(\mathbf{r})$ at the $n_{\tilde{n}} = n_f$ nodes of $\tilde{\mathcal{G}}$.

Faraday's equation and Ampère-Maxwell equation can be exactly written in terms of the introduced dofs. A stable second order discretization of constitutive equations can be achieved as follows. Let it be assumed that $\mathcal{G}$ is composed of four congruent squares of edge length 1 and that $\tilde{\mathcal{G}}$ is determined by the ξ coordinate, as shown in Fig. 1. The general case in which this simple pair of dual grids is independently scaled along the x and y directions and assembled to form non-uniform pairs of larger Cartesian dual grids follow from this case.

Let $\mathbf{w}_e^i(\mathbf{r})$, with $i = 1, \dots n_e = 12$, be a set of basis functions allowing to exactly reconstruct any affine field $\mathbf{E}(\mathbf{r})$ from its circulations e_i and let $\mathbf{w}_f^i(\mathbf{r})$, with $i = 1, \dots n_f = 4$, be a set of basis functions allowing to exactly reconstruct any affine field $\mathbf{B}(\mathbf{r})$ from its fluxes b_i. Referring to Fig. 1, it can be assumed

$$\mathbf{w}_e^1(x, y) = \mathbf{a}_y\,(-x/2 + \mathcal{P}(x)/2)(1/2 - y), \tag{1}$$

$$\mathbf{w}_e^2(x, y) = \mathbf{a}_y\,(1 - \mathcal{P}(x))(1/2 - y), \tag{2}$$

$$\mathbf{w}_f^1(x, y) = \mathbf{a}_z\,(1/2 - x)(1/2 - y), \tag{3}$$

in which $\mathbf{a}_y$, $\mathbf{a}_z$ are unit vectors directed as the y and z axes and $\mathcal{P}(x)$ is any function such that

$$\mathcal{P}(0) = 0, \tag{4}$$

$$\mathcal{P}(1) = 1, \tag{5}$$

$$\mathcal{P}(-x) = \mathcal{P}(x). \tag{6}$$

The basis functions for the other edges and faces can be straightforwardly derived from (1)-(3) as a consequence of geometrical symmetry.

Let $\mathcal{F}_x$ by the vector space determined by the three functions $-x/2+\mathcal{P}(x)/2$, $1-\mathcal{P}(x)$ and $x/2+\mathcal{P}(x)/2$ and let $\mathcal{L}_x$ be a *linear functional* of functions $f(x)$ such that

$$\mathcal{L}_x\left(1\right)=2, \tag{7}$$

$$\mathcal{L}_x\left(f(x)\right)=0 \text{ if } f(x)=-f(-x), \tag{8}$$

$$\mathcal{L}_x\left(f^2(x)\right)\geq 0, \tag{9}$$

$$\mathcal{L}_x\left(f^2(x)\right)=0 \text{ and } f(x)\in\mathcal{F}_x \text{ implies } f(x)=0. \tag{10}$$

It is noted that $\mathcal{L}_x$ can be written in the form

$$\mathcal{L}_x\left(f(x)\right)=\int_{-1}^{1}\mathcal{Q}_x(x)f(x)\,dx$$

in which $\mathcal{Q}_x(x)$ is either a function or a generalized function. Let now $\mathcal{L}_x=\mathcal{L}_y$ and let the linear functional $\mathcal{L}=\mathcal{L}_x\circ\mathcal{L}_y$ be introduced for functions $g(x,y)$, so that

$$\mathcal{L}(g(x,y))=\int_{-1}^{1}\mathcal{Q}_x(x)dx\int_{-1}^{1}\mathcal{Q}_y(y)g(x,y)dy.$$

This functional is a second order approximation of the integration operator over Ω. The elements

$$m_{\boldsymbol{\varepsilon}}^{ij}=\mathcal{L}(\mathbf{w}_e^i(\mathbf{r})\cdot\bar{\bar{\boldsymbol{\varepsilon}}}(\mathbf{r})\mathbf{w}_e^j(\mathbf{r})),$$

of matrix $\mathbf{M}_{\boldsymbol{\varepsilon}}$, with $i,j=1,\ldots,n_e=12$, and the elements

$$m_{\boldsymbol{\nu}}^{ij}=\mathcal{L}(\mathbf{w}_f^i(\mathbf{r})\cdot\bar{\bar{\boldsymbol{\nu}}}(\mathbf{r})\mathbf{w}_f^j(\mathbf{r})),$$

of matrix $\mathbf{M}_{\boldsymbol{\nu}}$, with $i,j=1,\ldots,n_f=4$, can now be determined as functions of the electric permittivity $\bar{\bar{\boldsymbol{\varepsilon}}}(\mathbf{r})$ and magnetic reluctivity $\bar{\bar{\boldsymbol{\nu}}}(\mathbf{r})$, assumed to be symmetric, positive definite tensors. These expressions differ from those of the energetic approach [2] just in the substitution of the integration operator over Ω with $\mathcal{L}$. With proper choices of ξ, $\mathcal{P}$ and $\mathcal{Q}_x=\mathcal{Q}_y$, equations

$$\tilde{\boldsymbol{d}}=\mathbf{M}_{\boldsymbol{\varepsilon}}\boldsymbol{e}, \tag{11}$$

$$\tilde{\boldsymbol{h}}=\mathbf{M}_{\boldsymbol{\nu}}\boldsymbol{b}, \tag{12}$$

determine stable, second order discretizations of constitutive equations. In fact the following properties can be easily proven.

Property 1: Equations (11), (12) exactly *transforms the circulations of* $\mathbf{E}(\mathbf{r})$ *along the edges of* $\mathcal{G}$ *into the fluxes of* $\mathbf{D}(\mathbf{r})$ *through the edges of* $\tilde{\mathcal{G}}$ *and the fluxes of* $\mathbf{B}(\mathbf{r})$ *across the faces of* $\mathcal{G}$ *into the components of* $\mathbf{H}(\mathbf{r})$ *at the nodes of* $\tilde{\mathcal{G}}$*, for all* affine *fields, if and only if it is*

$$\xi=(\sqrt{5}-1)/2, \tag{13}$$

$$\mathcal{L}_x(x^2)=\xi, \tag{14}$$

$$\mathcal{L}_x(\mathcal{P}(x)/2)=1-\xi. \tag{15}$$

Property 2: Matrices $\mathbf{M}_{\boldsymbol{\varepsilon}}$, $\mathbf{M}_{\boldsymbol{\nu}}$ are symmetric, positive definite.

Infinite choices of $\mathcal{P}$ and $\mathcal{Q}_x=\mathcal{Q}_y$ can be made in such a way that all properties (4)-(10) and (13)-(15) are satisfied. A simple choice is

$$\mathcal{P}(x)=|x|, \tag{16}$$

$$\mathcal{Q}_x(x)=c_{-\eta}\delta(x+\eta)+c_0\delta(x)+c_\eta\delta(x-\eta), \tag{17}$$

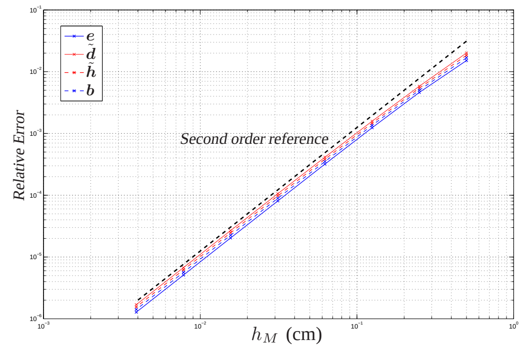

Figure 2: Relative error in maximum norm of dofs $\boldsymbol{e}$, $\boldsymbol{b}$, $\tilde{\boldsymbol{d}}$, $\tilde{\boldsymbol{h}}$, with respect to h_M.

in which δ is Dirac's delta and

$$\eta=(1+\xi)/2,$$

$$c_0=6-8\xi,$$

$$c_{-\eta}=c_\eta=4\xi-2.$$

It can be noted that in all situations the dual edges define *golden sections* of the primal edges, as established by (13). It can also be observed that the same set of conditions (7)-(10) and (13)-(15) leads to stable second order discretizations of both the electric and magnetic constitutive equations. Lastly, it is noted that it *cannot* be assumed $\mathcal{Q}_x=\mathcal{Q}_y=1$, since (14) would *not* be satisfied. As a result the integration operator over Ω used in the energetic approach [2] for achieving stable first order discretizations of constitutive equations *cannot* be employed also for stable second order discretizations. Conversely its second order approximation $\mathcal{L}$ *can* be used.

III. Numerical Results

A TM_{10} is injected into a section of rectangular waveguide terminated by a PEC. The discretization (16), (17) is applied to *non-uniform* Cartesian grids with decreasing maximum diameters h_M. As expected, the relative errors of *all* dofs $\boldsymbol{e}$, $\boldsymbol{b}$, $\tilde{\boldsymbol{d}}$, $\tilde{\boldsymbol{h}}$ exhibit second order convergence, as shown in Fig. 2.

IV. Conclusions

The full paper will include complete proofs and more details of the numerical analysis.

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