Asymptotic Boundary Conditions for Finite Element Analysis of 2D and 3D Electrical Field Problems

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Abstract—**This paper presents asymptotic boundary conditions (ABCs) suitable for the finite element modeling of 2D and 3D electrical field problem with open boundaries. The general form of the solution to Laplace's equation is used to derive the closed-form expressions for the *N*th order ABCs on circular, elliptical and spherical boundaries. To the best knowledge of the authors of this paper, the expressions have not yet been reported in the literature. The 1st and 2nd order ABCs, which can be easily implemented into existing finite element codes, are discussed in details, also for an arbitrary shape of the finite element region in 2D and for box-shaped boundaries in 3D. Implementation of the ABCs into commercial finite element software COMSOL Multiphysics is presented. Numerical examples are given.**

Index Terms—**Asymptotic boundary conditions, finite element method, open boundary problems, static fields.**

I. Introduction

Many 2D and 3D electrical field problems can be considered as being of the exterior form, that is the problem domain is unbounded. Since the finite element method is a finite domain method, special techniques must be employed when the solution domain is infinite. Over the last three decades various methods of analysis for the open boundary static and quasistatic electromagnetic field problems have been investigated [1], [2], [3], [4], [5], [6], [7]. Among the methods, asymptotic boundary conditions (ABCs) seem to be very attractive from the numerical point of view. In the present paper we discuss different aspects of the ABCs for the finite element analysis of 2D and 3D open boundary electrical field problems. General expressions for the *N*th order ABCs on circular, elliptical and spherical boundaries and numerical examples are given.

II. Asymptotic Boundary Conditions on Circular, Elliptical and Spherical Boundaries

To solve elliptic boundary value problems in an infinite domain by the finite element method, it is normal to divide the unbounded domain by an artificial boundary Γ into an interior region R_i (where sources, heterogeneities, anisotropies, etc. may exist) and a residual, uniform region R_e. When using the finite element method in R_i, some boundary conditions must be imposed on the artificial boundary Γ. The boundary conditions (called the ABCs) should mimic the behavior of the unknown potential V at infinity and give reasonably accurate results in the interior region R_i. The potential V in the exterior region R_e (and in the outermost part of R_i) satisfies the Laplace equation:

$$\nabla^2 V = 0 \tag{1}$$

The general solutions to (1), if the potential tends to zero at infinity, can be expressed as:

2D, polar coordinates (r, φ)

$$V(r,\varphi) = \sum_{n=1}^{\infty} r^{-n} F_{1n}(\varphi) \tag{2}$$

2D, elliptical coordinates (η, ψ)

$$V(\eta,\psi) = \sum_{n=1}^{\infty} \exp(-n\eta) F_{2n}(\psi) \tag{3}$$

3D, spherical coordinates (R, θ, φ)

$$V(R,\theta,\varphi) = \sum_{n=1}^{\infty} R^{-n} F_{3n}(\theta,\varphi) \tag{4}$$

The solutions (2), (3) and (4) can be used to obtain ABCs on the artificial boundary Γ. After some algebra, we have found general expressions for the *N*th order ABCs on circular, elliptical and spherical boundaries. They are as follows:

2D, circular boundary r = d

$$\frac{\partial^N V(d,\varphi)}{\partial r^N} + \sum_{m=1}^{N} \frac{\alpha_m^{(N)}}{d^{N-m+1}} \frac{\partial^{m-1} V(d,\varphi)}{\partial r^{m-1}} = O\left(d^{-2N-1}\right) \tag{5}$$

2D, elliptical boundary η = η₀

$$\frac{\partial^N V(\eta_0,\psi)}{\partial \eta^N} + \sum_{m=1}^{N} \beta_m^{(N)} \frac{\partial^{m-1} V(\eta_0,\psi)}{\partial \eta^{m-1}} = O\{\exp[-(N+1)\eta_0]\} \tag{6}$$

3D, spherical boundary R = d

$$\frac{\partial^N V(d,\theta,\varphi)}{\partial R^N} + \sum_{m=1}^{N} \frac{\alpha_m^{(N)}}{d^{N-m+1}} \frac{\partial^{m-1} V(d,\theta,\varphi)}{\partial R^{m-1}} = O\left(d^{-2N-1}\right) \tag{7}$$

where: $\alpha_m^{(N)} = \binom{N}{m-1}\frac{N!}{(m-1)!}$, $\beta_m^{(N)} = \begin{bmatrix} N+1 \\ m \end{bmatrix}$, $\frac{\partial^0 V}{\partial \tau^0} = V$.

The coefficients $\alpha_m^{(N)}$ are known as coefficients of the Laguerre polynomials, whereas the coefficients $\beta_m^{(N)}$ are the unsigned Stirling numbers of the first kind and can be calculated by the recurrence relation:

$$\begin{bmatrix} N+1 \\ m \end{bmatrix} = N \begin{bmatrix} N \\ m \end{bmatrix} + \begin{bmatrix} N \\ m-1 \end{bmatrix},$$

for m > 0, with the initial conditions:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1, \quad \begin{bmatrix} N \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1.$$

To the best knowledge of the authors of this paper, the ABCs (5), (6) and (7) have not yet been reported in the literature in such forms. No doubt, that the closed-form expressions for the *N*th order ABCs are very important from the theoretical point of view, however, in fact only the first and second order ABCs can be relatively easily implemented into existing finite element codes. The boundary contribution in the finite element method enters into a line integral representation over the outer boundary Γ, where the integrand is a product of the weighting function (shape function) and the normal derivative of the unknown function V. Hence, the ABCs need to be imposed on the normal derivative of V. In a full version of the paper the first and second order ABCs will be discussed in details.

III. Asymptotic Boundary Conditions for Box-Shaped Boundaries

The choice of the artificial boundary to be a circle (ellipse) in two dimensions or a sphere in three dimensions enabled simple derivation of the ABCs. However, such boundaries are uneconomical for problems with large aspect ratios. In a full version of the paper an arbitrary shape of the two-dimensional finite element region will be discussed. In the present digest the ABCs for box-shaped (3D) boundaries are considered. It is necessary to derive the appropriate normal derivative expressions for the different faces of the box representing the outer boundary Γ. Using the relations between the Cartesian and spherical coordinates, we have found the relevant ABCs:

3D, 1st order ABC, x = const = d

$$\frac{\partial V}{\partial x} = -\frac{1}{d}V - \frac{y}{d}\frac{\partial V}{\partial y} - \frac{z}{d}\frac{\partial V}{\partial z} \tag{8}$$

3D, 2nd order ABC, x = const = d

$$\frac{\partial V}{\partial x} = -\frac{1}{2d}V + \frac{d^2+y^2}{4d}\frac{\partial^2 V}{\partial y^2} + \frac{d^2+z^2}{4d}\frac{\partial^2 V}{\partial z^2} + \frac{zy}{2d}\frac{\partial^2 V}{\partial y \partial z} \tag{9}$$

with the conditions on the other faces obtained by replacing x with y and y with x for y = const = d, and x with z and z with x for z = const = d (in (9) we have used the 1st order ABC to approximate the terms $\partial^2 V/\partial x \partial y$ and $\partial^2 V/\partial x \partial z$; condition (9) differs from that given in [2]).

An implementation of ABCs in FEM program is relatively simple when the code of the program is available. However, commercial programs are usually closed-source software packages. Fortunately, in COMSOL Multiphysics it is enough to choose so called *surface charge* boundary conditions, $\boldsymbol{n} \cdot \boldsymbol{D} = \rho_s$, in the boundary conditions section. In the section modifications of typical boundary conditions are possible in an easy way by introducing user's own formulas.

IV. Numerical Example

A simple 2D problem with a known closed-form solution was considered. This was the scalar potential distribution due to a dipole consisting of two lines with charge densities $\pm\lambda$ on the x-axis at positions $x = \pm a$, respectively. The problem was solved numerically with $a = 0.5$ and $\lambda = \varepsilon_0$. This example was chosen in [1] for testing infinite elements. Due to symmetry, the domain of solution was only one-quarter of the plane. Solutions of the problem by different techniques are compared qualitatively in Fig. 1 (equipotential lines are shown in each case). To check if the methods work well, a strange form of the finite element region with cut-out was chosen for calculation.

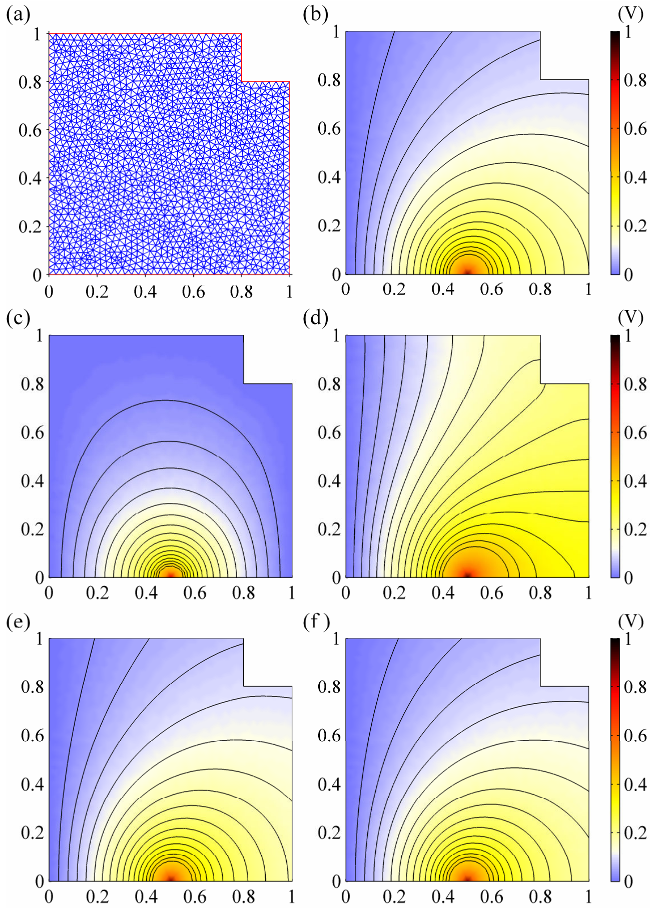

Fig. 1. Test problem: a) finite element mesh, b) analytical solution, c) zero Dirichlet boundary condition, d) zero Neumann boundary condition, e) infinite elements [1], f) 1st order ABC

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