Convergence Characteristics of Preconditioned Linear Solvers Based on Minimum Residual for Complex Symmetric Linear Systems

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***Abstract*—IC-COCG (Conjugate Orthogonal Conjugate Gradient method with Incomplete Cholesky decomposition) method is widely used to solve complex symmetric systems derived from frequency domain edge-based finite element method. This paper shows the effectiveness of COCR (Conjugate *A*-Orthogonal Conjugate Residual) and MRTR (Minimized Residual Method based on the Three-term Recurrence formula of CG-type) methods with IC preconditioner.**

***Index Terms*—Convergence, finite element methods, iterative methods, linear systems, symmetric matrices.**

I. Introduction

The electromagnetic field analysis using an edge-based finite element method requires solving linear systems. The fast computation of linear systems is essential to reduce the elapsed time in using finite element analysis. Generally, ICCG method is widely used to real symmetric linear systems in time domain. Recently, we make the effectiveness of preconditioned MRTR method clear [1]. It is shown that the convergence characteristics of MRTR method based on the minimum residual are better than that of CG method.

Similarly, MRTR method for solving complex symmetric linear system (COMRTR), which is mathematically equivalent to COCR method, is presented [3], [4]. However, the performance of COMRTR to the complex symmetric matrix is largely unknown. Then, this paper shows the superiority of preconditioned COMRTR method with the split preconditioners, which are shifted Incomplete Cholesky (IC) and Complex Shifted IC (CSIC) [5] decomposition.

II. Some Split Preconditioners

A. IC Preconditioner

A complex symmetric sparse linear system can be defined as follows:

$$A\boldsymbol{x} = \boldsymbol{b}\,, \tag{1}$$

where A is a complex symmetric matrix. Now, suppose that diagonal scaling has already been applied to (1). Resulting all real and imaginary parts in diagonal components of A become 1.0 and 0.0, respectively.

IC factorization is performed as follows:

$$l_{ii} = \gamma\, a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 d_{kk}\,, \tag{2}$$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} d_{kk} \qquad (i \neq j)\,, \tag{3}$$

$$d_{ii} = 1 / l_{ii}\,, \tag{4}$$

where a_{ij}, l_{ij} and d_{ii} are components of A, $\hat{L}$ and $\hat{D}$, respectively, and γ is the shift parameter (real number). γ is determined by the following steps: 1) set $\gamma = 1.05$, 2) perform IC factorization, 3) if all real parts of diagonal components (Re$\{l_{ii}\}$) become positive, IC factorization stops; otherwise, return to step 1, add 0.05 to γ, and iterate steps 1) – 3).

B. CSIC Preconditioner

CSIC preconditioner uses the shift parameter, which is implemented to the imaginary parts of diagonal components l_{ii} as follows:

$$l_{ii} = (a_{ii} + \mathrm{j}\alpha) - \sum_{k=1}^{i-1} l_{ik}^2 d_{kk}\,, \tag{5}$$

where j is an imaginary unit and α is the shift parameter (real number). l_{ij} and d_{ii} are calculated by (3) and (4), respectively.

III. Analysis Model

Table 1 lists the specifications of the analysis model. The magnetic shielding is divided into four layers and its thickness is 1 mm. The induction heating (IH) cooker model contains frying pan, which is discretized by ten layers elements with thickness of 2 mm.

The iterative process is terminated under the convergence criterion $\|\boldsymbol{r}_k\|_2 / \|\boldsymbol{b}\|_2 < \varepsilon$, where $\|\boldsymbol{r}_k\|_2$ is a 2-norm of residual and ε is set to 10^{-7}. All problems are computed by using a single thread of the PC having an Intel Core i7 3770K / 4.5 GHz over-clocked CPU and 32 GB RAM.

IV. Numerical Experiments

A. Effectiveness of Shift Parameter

Fig. 1 shows the number of iterations versus shift parameter in box shield model. The number of IC-COCG iterations is minimum at γ = 1.10. On the other hand, the number of CSIC-COCG iterations decreases exponentially. However, CSIC-COCG iterations tends to be larger than that of IC- COCG iterations.

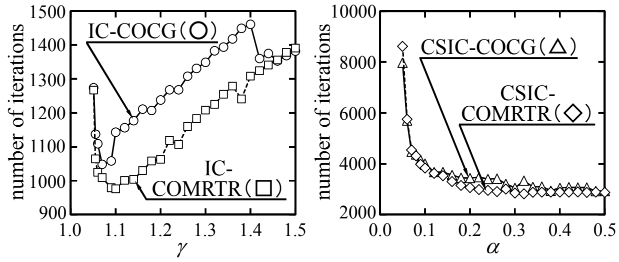
Fig. 2 shows the number of iterations based on shift parameter in IH cooker model. The number of CSIC-COCG iterations is larger than that of IC-COCG iterations.

B. Performance of the Preconditioned Linear Solvers

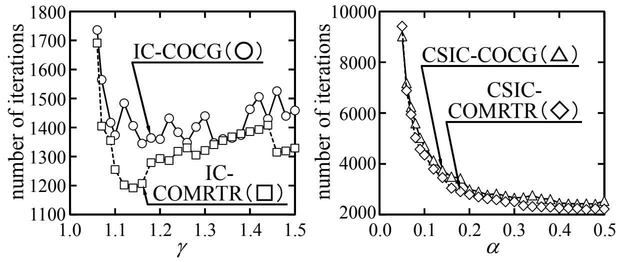
Fig. 3 shows the convergence characteristics. The characteristics of the preconditioned COMRTR method are almost consistent with those of preconditioned COCR method. IC preconditioners mostly improve the convergence

TABLE I
ANALYZED CONDITIONS

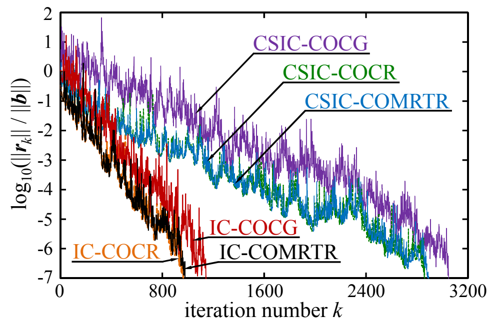
analysis model	formul.	discret.	no. of elements	DoF	nonzero
Box shield model	$\boldsymbol{A}-\phi$	1st-hexa	67,980	206,427	3,740,594
IH cooker model	$\boldsymbol{A}-\phi$	1st-hexa	799,456	2,461,357	46,204,111



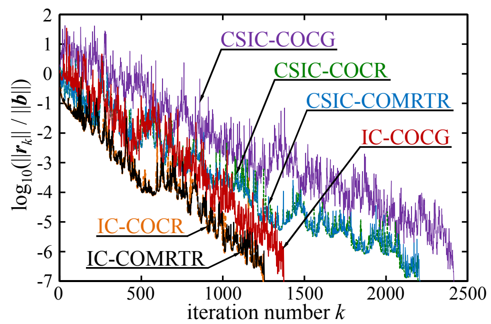
(a) IC preconditioner (b) CSIC preconditioner
Fig. 1. Effect of shift parameter on box shield model.



(a) IC preconditioner (b) CSIC preconditioner
Fig. 2. Effect of shift parameter on IH cooker model.



(a) box shield model



(b) IH cooker model

Fig. 3. Convergence characteristics of preconditioned linear solvers using the shift parameters: (a) $\gamma = 1.10$, $\alpha = 0.40$, (b) $\gamma = 1.10$, $\alpha = 0.45$.

characteristics by the comparison with CSIC preconditioners. The IC-COMRTR and IC-COCR methods converged faster than IC-COCG method.

Table II and III show the elapsed time of various solvers. The elapsed time of preconditioned COMRTR method becomes a little longer than that of preconditioned COCR method. This is caused by two following reasons: 1) the number of COMRTR iterations is larger than that of COCR iterations, 2) the computational costs of the inner product, the sum of vectors and a scalar-vector product in preconditioned COMRTR method is larger than that of preconditioned COCR method. IC-COCR and IC-COMRTR methods are effective in reducing the elapsed time.

Hereafter, we investigate various preconditioners such as symmetric SOR and so on. The effectiveness of preconditioned COMRTR is more discussed in the full paper.

TABLE II
ANALYZED RESULTS FOR THE BOX SHIELD MODEL

linear solver	precond.	total it.	elapsed time [s]
COCG	–	4,058 (3.55)	39.5 (1.53)
	IC	1,143 (1.00)	25.8 (1.00)
	CSIC	3,049 (2.66)	68.5 (2.65)
COCR	–	3,758 (3.28)	39.3 (1.52)
	IC	965 (0.84)	23.1 (0.89)
	CSIC	2,861 (2.50)	67.6 (2.62)
COMRTR	–	3,768 (3.29)	39.3 (1.52)
	IC	978 (0.85)	24.0 (0.93)
	CSIC	2,893 (2.53)	70.9 (2.74)

TABLE III
ANALYZED RESULTS FOR THE IH COOKER MODEL

linear solver	precond.	total it.	elapsed time [s]
COCG	–	4,275 (3.11)	552.0 (1.38)
	IC	1,374 (1.00)	399.3 (1.00)
	CSIC	2,416 (1.75)	693.4 (1.73)
COCR	–	3,466 (2.52)	480.8 (1.20)
	IC	1,249 (0.90)	381.2 (0.95)
	CSIC	2,198 (1.59)	666.1 (1.66)
COMRTR	–	3,470 (2.52)	486.9 (1.21)
	IC	1,255 (0.91)	392.6 (0.98)
	CSIC	2,209 (1.60)	693.3 (1.73)

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