Reduced Basis Generation for Maxwell's Equations by Rigorous Error Estimation

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***Abstract*—The Reduced Basis Method (RBM) generates low-order models of parametrized partial differential equations (PDEs) to allow for efficient evaluation of parametrized models in many-query and real-time contexts.**

We use the RBM to generate low order models of microscale semiconductor devices under variation of geometry and frequency. In particular, we focus on the efficient rigorous error estimation, which enables to generate low-order models with certified accuracy. We test and compare multiple techniques to compute lower bounds to the discrete stability constant, which is a challenging problem in the context of Maxwell's equations.

***Index Terms*—Reduced order systems, Finite element methods, Electromagnetic fields, Numerical analysis**

I. Introduction

The Reduced Basis Method (RBM) generates low order models for the efficient solution of parametrized partial differential equations (PDEs) in real-time and many-query scenarios. The RBM employs rigorous error estimators to perform the model reduction and to measure the quality of the reduced simulation. In recent years, the RBM has been developed to apply to a wide range of problems, of which [1] and the references therein, give an overview.

We address the use of the RBM in a time-harmonic electromagnetic problem, which exhibits parameter variation in geometry and frequency. We use the RBM in a 3D model of a coplanar waveguide, as an example of a microscale semiconductor structure. Recent work in parametric model order reduction (PMOR) within the electromagnetic regime uses multipoint expansion techniques ([2], [3] and [4]) and proper orthogonal decomposition (POD) ([5]). Geometric parameter variations are also investigated in [4].

Section 2 introduces the model and the constitutive equations. Section 3 states the main properties of the reduced basis model reduction, while section 4 gives details on the error estimation. Finally, section 5 gives performance indices of the model reduction.

II. Model Problem

As an example model, we consider the coplanar waveguide, depicted in Fig. 1. The model setup is contained in a shielded box with perfect electric conducting (PEC) boundary. We consider three perfectly conducting striplines as shown in the geometry. The system is excited at a discrete port and the output is taken at a discrete port on the opposite end of the middle stripline. These discrete ports are used to model input and output currents/voltages.

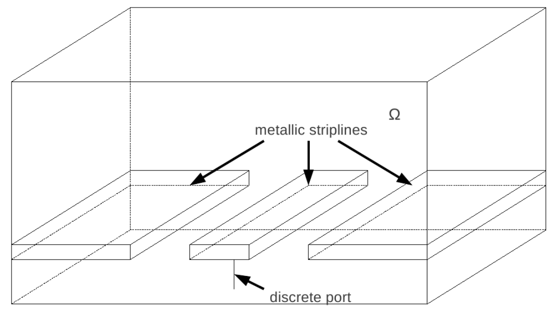

Figure 1: Geometry of coplanar waveguide.

We consider the second order time-harmonic formulation of Maxwell's equations in the electric field E

$$\nabla \times \mu^{-1} \nabla \times E + j\omega\sigma E - \omega^2 \epsilon E = j\omega J \quad \text{in } \Omega, \qquad (1)$$

subject to zero boundary conditions

$$E \times n = 0 \quad \text{on } \Gamma_{\text{PEC}}.$$

We use the weak formulation of (1) with sesquilinear form $a(\cdot,\cdot;\nu)$ and linear form $f(\cdot;\nu)$ as

$$a(E(\nu), w; \nu) = f(w; \nu) \quad \forall w \in X, \qquad (2)$$

where $\nu \in \mathcal{D} \subset \mathbb{R}^p$ denotes the parameter vector, $E(\nu)$ is the parameter-dependent electric field, w a test function and X the $H(\text{curl})$-conforming finite element space, discretized with Nédélec finite elements [6].

III. Reduced Basis Method for time-harmonic EM-problems

The aim of the RBM is to determine a low order space X_N of dimension N, which approximates the parametric manifold

$$M^\nu = \{E(\nu) | \nu \in \mathcal{D}\}$$

well. Given such a space X_N, it is possible to gain accurate approximations $E_N(\nu)$ to $E(\nu)$ by solving (2) in X_N

$$a(E_N(\nu), w_N; \nu) = f(w_N; \nu) \quad \forall w_N \in X_N, \qquad (3)$$

i.e. projecting (2) onto X_N.

An integral part in the model reduction are error estimators $\Delta_N(\nu)$, which give rigorous bounds to the approximation error in the discrete $H(\text{curl})$ norm.

Additionally, the RBM requires fast evaluations of the error estimator in the sense that the algorithmic complexity is independent of the discretisation size of the full model. For that, the RBM uses an affine decomposition [1] of the bilinear and linear form

$$a(E(\nu), w; \nu) = \sum_{q=1}^{Q_a} \Theta_a^q(\nu) a^q(E(\nu), w), \tag{4}$$

$$f(w; \nu) = \sum_{q=1}^{Q_f} \Theta^q(\nu) f_a^q(w).$$

Given a RB space $X_N = \{\zeta_1, \ldots, \zeta_N\}$, the bilinear forms $a^q(\zeta_i, \zeta_j)$ are precomputed and plugged into (4) to allow a fast assembly of the parameter-dependent system [1].

A. Geometric Parameters

To consider the linear combination of snapshots for different geometries, the PDE is transformed from the parameter-dependent domain $\Omega(\nu)$ to a parameter-independent reference domain $\Omega(\overline{\nu})$.

With a domain decomposition of $\Omega(\overline{\nu})$ into subdomains, such that each subdomain under consideration can be found as an affine transformation of the respective reference subdomain, then the affine decomposition (4) can be derived from the affine geometry transformations.

IV. Error Estimation

The error estimator for the electric field is given by

$$\Delta_N(\nu) = \frac{\|r(\cdot;\nu)\|_{X'}}{\beta_{LB}(\nu)},$$

with $\|r(\cdot;\nu)\|_{X'}$ the dual norm of the residual with respect to the full scale discretised problem and $\beta_{LB}(\nu)$ a lower bound to the discrete inf-sup stability constant. The error estimator gives rigorous bounds in the sense that

$$\|E(\nu) - E_N(\nu)\|_X \leq \Delta_N(\nu),$$

where X denotes the discrete $H(\text{curl})$ norm.

The computation of a lower bound to the discrete stability constant $\beta_{LB}(\nu)$ is the most time-consuming part in rigorous error estimation, especially when applied to Maxwell's equations, see [7].

We investigate the performance of different techniques for obtaining lower bounds to the discrete stability constant, such as successive constraint methods [7], the methods derived in [9] and interpolation methods. For the example model of a coplanar waveguide, the discrete stability constant is shown in Fig. 2 under variation of frequency and the width of the middle stripline.

V. Numerical Results

The full simulation has been performed with the finite element package FEniCS using a discretization with first order Nédélec finite elements. For our numerical experiments, we used a discretization size of $52'134$ degrees of freedom. The selected parameter variation is the frequency, which varies in

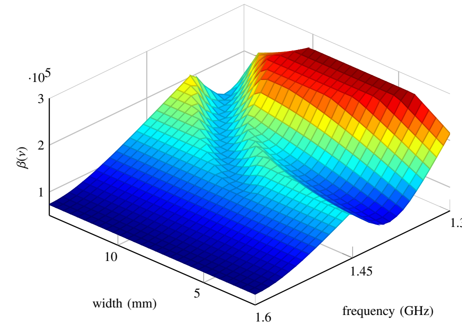

Figure 2: Stability constant plotted over variation of frequency and geometry.

$[0.6, 3.0]$ GHz and the width of the middle stripline, which varies in $[2.0, 14.0]$ mm.

We observed exponential convergence speed in our experiments, in particular a relative error of less than 1% can be achieved using 59 basis functions, which is a significant reduction from the original number of degrees of freedom. See [8] for more details on the model reduction performance indices.

The RBM typically employs a successive constraint method (SCM) to obtain lower bounds to the discrete stability constant. In particular the application to Maxwell's equations shows slow convergence, see [7]. We test and compare different variants of the SCM and apply certain modifications, like introducing randomized constraints into the SCM which yields speedup factors of 3 to 4.

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