

An Alternative Way to Impose Essential Boundary Conditions in EFG Method

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Abstract— The Element Free Galerkin method is very effective for solving electrostatic problems. However it leads to shape functions which do not satisfy the Kronecher delta property. So it is necessary to use a technique to impose essential boundary conditions. In this work we use the Element Free Galerkin method with the interpolating shape functions. So the Kronecher delta property is satisfied and the essential boundary conditions can be enforced directly in the discrete system. The technique is applied in the analysis of a cylindrical coaxial capacitor filled with two different concentric dielectric materials.

Index Terms— Numerical simulation, Electromagnetic modeling and Electrostatic analysis.

I. INTRODUCTION

The analysis of electromagnetic phenomena, in general, involves the solution of partial differential equations (PDE), which, in most cases, require the use of numerical methods. One of the most widely used numerical methods for solving this kind of problem is the Finite-Element Method (FEM). It presents characteristics that permit taking into account different kinds of materials and modeling geometrically complex domains. However, a good quality mesh is needed in order to produce accurate results.

In the last twenty years, a new class of numerical methods for solving PDE, the Meshless Methods, has been developed. Though most of these methods originated from mechanical problems, they have been extended for solving electromagnetic ones. The Meshless technique eliminates the mesh generation by requiring the spatial domain to be represented by a set of nodes, rather than a mesh. The nodes are not connected to each other and there is no relationship between them. This characteristic offers great advantages over the mesh based methods mainly for modeling complex geometries and moving bodies and boundaries [1]-[2].

Lately several different Meshless Methods have been proposed [1]. Among these methods, one of the most promising is the Element Free Galerkin Method (EFG) [3]. It presents good convergence rates, ease of creating discretization and independence of integration in the weak form. This Method is very popular and its application has been extended to different classes of problems. In Meshless Methods, the shape functions do not have analytical expressions, being built numerically during the solution of the problem, and there are many different ways to build them. The EFG uses Moving Least Square Method (MLSM) to construct the shape functions [1]. These functions are used to build a finite dimensional subspace and the Galerkin procedure is employed to obtain an approximation of the unknown function. However the MLSM provides shape functions which

do not satisfy the Kronecher delta property, thus, it is necessary to use a technique to prescribe the Essential Boundary Conditions (EBC). One of the most used techniques for doing this is the Lagrange Multipliers Method (LMM), which leads to accurate results, however it is necessary to perform a integral over the boundary where the EBC should be enforced. Since the Lagrange Multipliers are unknown functions, the total number of unknown functions of the entire system is increased. However, it is possible to do an adaptation in the MLSM so that the shape functions satisfy the Kronecher delta property. This procedure is known as the Interloping Moving Least Square Method (IMLS) [4]-[5] and is performed using a singular window function, thus the EBC can be imposed directly into the discrete system.

In this paper, we present the approach of EFG using both the MLSM and IMLS applied to the analyses of an electrostatic problem, a cylindrical coaxial capacitor filled with two concentric dielectric materials. We compare the numerical results with the analytical solution.

II. WEAK FORM AND LINEAR SYSTEM

The capacitor studied in this work, illustrated in Fig. 1, is described by the strong form defined by the Laplace equation:

$$\nabla \cdot (\nabla V) = 0, \quad (1)$$

where V is the electrostatic potential, Γ_i and Γ_e are the Dirichlet boundaries where V should be imposed and ε_1 and ε_2 are the electric permittivity on the dielectric domains Ω_1 and Ω_2 , respectively.

Starting from the functional stationary point relative to variations in V , the following weak form is obtained [6]:

$$\int_{\Omega} \varepsilon \nabla V \cdot \nabla u \, d\Omega = 0 \quad \forall u \in H^1(\Omega), \quad (2)$$

where u is the weighting function, Ω is the problem domain, H^1 is the Sobolev subspace and V should be determined.

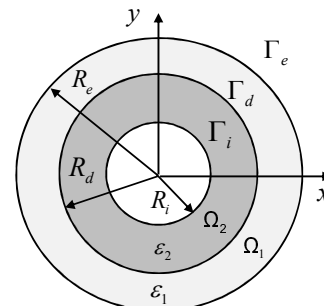


Fig. 1. Cylindrical coaxial capacitor

The Galerkin method is used to obtain discrete equations of the weak form using the following approximations [1]:

$$u(\mathbf{x}) \approx \sum_{i=1}^N \Phi_i(\mathbf{x}) u_i, \quad (3)$$

$$V(\mathbf{x}) \approx \sum_{i=1}^N \Phi_i(\mathbf{x}) V_i, \quad (4)$$

where $\mathbf{x} = (x, y)$ and Φ_i is the EFG shape function. So the linear system $[K][V] = 0$ is obtained, where

$$K_{ij} = \int_{\Omega} \varepsilon \nabla \Phi_i \cdot \nabla \Phi_j \, d\Omega, \quad (5)$$

III. EFG SHAPE FUNCTION

The EFG shape functions are constructed using the MLSM. So, minimizing a weighted discrete error norm, the shape function for each node, I , is:

$$\Phi_I(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}_I(\mathbf{x}), \quad (6)$$

where

$$\mathbf{p}^T(\mathbf{x}) = [1, x, y], \quad (7)$$

$$\mathbf{A}^{-1}(\mathbf{x}) = \sum_{I=1}^n \omega(\mathbf{x} - \mathbf{x}_I) \mathbf{p}(\mathbf{x}_I) \mathbf{p}^T(\mathbf{x}_I), \quad (8)$$

$$\mathbf{B}_I(\mathbf{x}) = \omega(\mathbf{x} - \mathbf{x}_I) \mathbf{p}(\mathbf{x}_I), \quad (9)$$

The shape functions are constructed by finding the nodes contained in the support of the weight function, $\omega(\mathbf{x} - \mathbf{x}_I)$. The connectivity of the nodes is due to various influence domains overlapping each other. Φ_I can be chosen so that it satisfies the Kronecher delta property, so, a singular window function is used [4]-[5]. This kind of function is infinite on node and tends instantaneously to zero on the other points. Thus, the EBC can be imposed directly into the discrete system and the shape function can interpolate the desired function. In this work the following singular function is used:

$$\omega(r) = 1/(r^n + \beta^n), \quad (10)$$

where β is a constant whose value must be small enough to ensure no singularity, n is a constant whose value is adjusted to improve the accuracy and [1]

$$r = \sqrt{\left[\frac{(x - x_I)}{d} \right]^2 + \left[\frac{(y - y_I)}{d} \right]^2}, \quad (11)$$

where $d = ck$ is the support radius of circular influence domain of each node, c is a proportionality constant ($1.5 \leq c \leq 4$) and k depends on the node distribution [1].

IV. NUMERICAL RESULTS

The accuracy of the numerical results was verified against analytical solutions using error in the following Lesbegue norm:

$$\|e(x)\|_{L^2(\Omega)} = \int_{\Omega} e(x)^2 \, d\Omega, \quad (13)$$

with $e(x)$ defined by

$$e(x) = X_{meshless}(x) - X_{analytical}(x), \quad (14)$$

where X is electric potential, V , or electric field, \mathbf{E} .

For the cylindrical coaxial capacitor illustrated in Fig. 1 it was considered: $R_i = 1$ m, $R_d = 1.5$ m, $R_e = 2$ m, $\varepsilon_1 = \varepsilon_0$, $\varepsilon_2 = 4\varepsilon_0$, $V = 1V$ on Γ_i and $V = 2V$ on Γ_e . The analysis was performed using 386 nodes distributed in a cylindrical form, 3053 Gauss points distributed in a rectangular form, $c = 1.5$ and the discontinuity at R_d was considered using the visibility criterion. The electric potential obtained on a radial line, ρ , from R_i to R_e is shown in Fig. 2. A good conformity can be observed between analytical and IMLS numerical solution. The error in the Lesbegue norm is presented in Tab. 1.

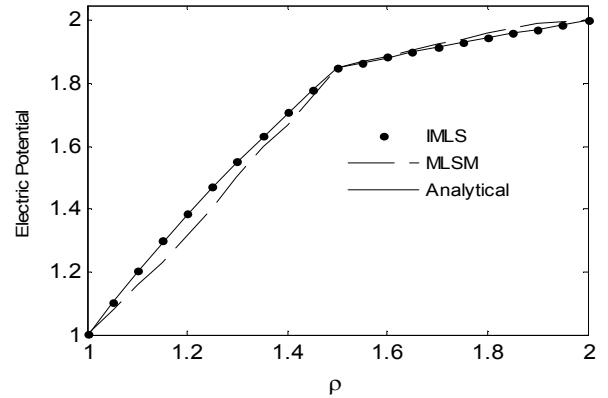


Fig. 2. Electric potential

TABLE I
ERROR IN THE LESBEGUE NORM

	MLSM	IMLS
Electric potential (V)	0.0758	0.00374
Electric field (V/m)	1.188	0.0498

The numerical results obtained with IMLS required 10% less time than MLSM and its precision is better as shown in Fig 2 and Tab. 1. These factors demonstrate the accuracy and efficiency of investigated numerical technique.

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