Convergence of a Stabilized Subpixel Smoothing Scheme for the Finite Integration Technique

Rolf Schuhmann
TU Berlin, Fachgebiet Theoretische Elektrotechnik
Einsteinufer 17, EN-2, 10587 Berlin, Germany
schuhmann@tet.tu-berlin.de

***Abstract*—A recently proposed subpixel smoothing scheme for the accurate modeling of non-conforming dielectric interfaces in 3D field simulations is investigated. The modified material matrix is formulated in terms of the methodology of the Finite Integration Technique and used in frequency and time domain. In order to ensure the longtime stability of the time domain scheme the material matrix has to be symmetrized, and some numerical results for a test problem are presented and discussed.**

***Index Terms*—Finite Difference Methods, Convergence, Numerical Stability.**

I. Introduction

The accurate treatment of oblique interface planes in a rectangular (Cartesian) computational mesh has for long been one of the major challenges in Finite Difference (FD) and Finite Integration (FIT) schemes. The well-known staircase modeling of the original algorithm may lead to a considerable and in most cases non-smooth error behavior even for well-resolved meshes. Some advanced techniques presented so far can roughly be divided into two classes. The first one is the extension of FD and FIT schemes to more general, body-conforming meshes such as triangular grids in 2D, non-orthogonal coordinate grids, or fully unstructured grids (see, e.g., references in [1]). These methods do not show the geometric modeling error any more, however at the price of being less efficient, e.g. due to their implicit nature in time domain. The 2nd class of extended methods stick to the Yee mesh, but allow the use of non-homogeneous material fillings within single cells ('subcell methods'). This includes the simple triangular filling scheme [2] as well as the conformal algorithms [3][4] for interfaces to perfect conductors.

Recently [5] a subcell-technique was proposed which is also able to handle dielectric interfaces within one cell. In this method, referred to as 'subpixel smoothing', an effective, anisotropic layer is constructed such that tangential and normal field components at the interface are handled by different averaging schemes. In [6],[7],[8] it was recognized that this method may suffer from late-time instabilities, depending on how the anisotropy is implemented in the FDTD code.

We review this method in the FIT notation and analyze the stability issues. Following some earlier ideas in [9],[10], the anisotropy is implemented in some variants and the effect on the method's accuracy is investigated.

II. Subpixel Smoothing

A. Standard FIT with Classical Material Operators

The main idea of the so-called subpixel smoothing is based on a well-known result from the classical Finite Integration Technique (of correlated Finite Difference approaches). FIT [2] uses a pair of staggered grids G (primary) and $\tilde{G}$ (dual), restricted her to the Cartesian case for simplicity. The degrees of freedom are the so-called integral state variables, defined as integrals over edges $L_i, \tilde{L}_i$ and facets $A_j, \tilde{A}_j$ of the primary grid G and the dual grid $\tilde{G}$, respectively:

$$\widehat{e}_i = \int_{L_i} \vec{E} \cdot d\vec{s}\,, \qquad \widehat{\widehat{d}}_i = \int_{\tilde{A}_i} \vec{D} \cdot d\vec{A}\,, \tag{1}$$

$$\widehat{\widehat{j}}_i = \int_{\tilde{A}_i} \vec{J} \cdot d\vec{A}\,, \quad \widehat{h}_j = \int_{\tilde{L}_j} \vec{H} \cdot d\vec{s}\,, \quad \widehat{\widehat{b}}_j = \int_{A_j} \vec{B} \cdot d\vec{A}\,. \tag{2}$$

The Maxwell's grid equations can be written as

$$\mathbf{C}\widehat{\mathbf{e}} = -\partial_t \widehat{\widehat{\mathbf{b}}}, \qquad \mathbf{C}^T \widehat{\mathbf{h}} = \partial_t \widehat{\widehat{\mathbf{d}}} + \widehat{\widehat{\mathbf{j}}}\,, \tag{3}$$

with the curl-operator $\mathbf{C}$. The material relations $\widehat{\widehat{\mathbf{d}}} = \mathbf{M}_\varepsilon \widehat{\mathbf{e}}$, $\widehat{\widehat{\mathbf{j}}} = \mathbf{M}_\kappa \widehat{\mathbf{e}}$, $\widehat{\widehat{\mathbf{b}}} = \mathbf{M}_\mu \widehat{\mathbf{h}}$ will be discussed in detail below.

Already in the very standard implementation – with homogeneously filled, Cartesian mesh cells – an averaging of permittivities has to take place at material interfaces. The material relation $\widehat{\widehat{d}}_n = \mathbf{M}_{\varepsilon,nn}\, \widehat{e}_n$ for a simple component correlates a grid voltage at a primary edge to the electric grid flux at the corresponding dual face. The classical formula

$$\mathbf{M}_{\varepsilon,nn} = \tilde{A}_n \left\langle \varepsilon \right\rangle L_n^{-1} \tag{4}$$

defines a one-to-one relation and thus a matrix operator in diagonal form. At 'standard' material interfaces, where a tangential discrete electric component is allocated, $\left\langle \varepsilon \right\rangle$ has to be calculated as an area-based averaging of the permittivities.

It has been shown (see, e.g., [11]) that the area-based averaging of ε can be replaced by an averaging $\left\langle \varepsilon^{-1} \right\rangle$ of the inverse permittivity for subcell situations where the normal dielectric flux density is the continuous component in the grid. None of the two alone, however, is sufficient for the general sub-cell case in Fig. 1 (left), where the orientation of the fields with respect to the interface plane is not fixed a-priori.

B. Effective Tensorial Permittivity

In order to combine the standard and inverse averaging formulas, an effective permittivity has been proposed in [5]. Using a geometric projector $\mathbf{P}$ with $P_{ij} = n_i n_j$ (derived locally from the interface normal), the electric field can be split into its tangential and normal components. Each of them is treated by a separate averaging formula which leads to an effective, tensorial permittivity with a non-diagonal anisotropy:

$$\underline{\underline{\varepsilon}}^{-1} = \mathbf{P}\left\langle \varepsilon^{-1} \right\rangle + (\mathbf{I}-\mathbf{P})\left\langle \varepsilon \right\rangle^{-1}. \qquad (5)$$

This formula has been implemented in a number of references, e.g. [1],[5],[6],[8], and proven to show a considerably improved accuracy for general interfaces. Note that the permittivity is defined in its inverse form to fit to common frequency and time domain formulations.

C. FIT / FDTD Implementation and Stability

The definition of the effective permittivity alone is not yet sufficient, since its strong anisotropy goes beyond the classical FIT/FDTD implementation. Two issues have to be solved: The first, simpler one is the fact that the three electric field (or flux) components are not defined at the same location in the staggered grid, but require some interpolation technique. This leads to non-diagonal material matrices with up to 8 off-diagonal entries, comparable to [9]. However – the 2nd problem – such an interpolation may lead to long-time instability if the symmetry of these matrices is spoiled. Without special care this will always happen, especially – considering the full material formula (4) – in case of non-uniform grids.

Similar problems have been solved earlier in the context of FIT, e.g. for non-orthogonal grids [9] or gyrotropic materials [10]. Note that the symmetry issue has also been recognized in [6],[7],[8], and some remedies are discussed there.

III. Symmetrization and Numerical Results

The extended, non-diagonal permittivity matrix (see Fig. 2, left) has been implemented and used for eigenmode computations in a resonator with a dielectric cylinder. (Quasi-2D TE modes, featuring both normal and tangential field components at the interface, in a full 3D implementation). The transversal size of the cavity (see Fig. 1, right) is 2m x 2m, and the radius of the cylinder (with ε_r=10) is R=0.5m. The mode considered here shows a TE_{100}-like field pattern and a resonance frequency of f_{ref} = 57.14 MHz (reference solution from a high-resolution, 2nd order Finite Element simulation.)

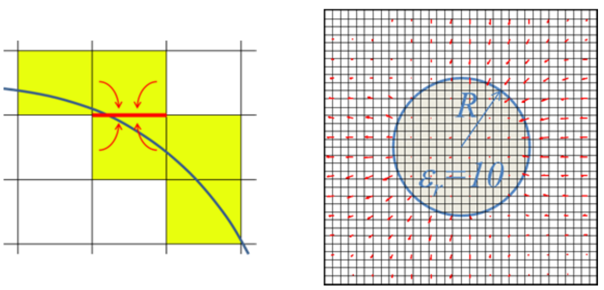

Fig. 1. Basic interpolation scheme, test structure with 35x35 grid (cut plane).

The problem has been solved both in frequency domain (where a non-symmetric material operator may be used without stability problems) and time domain. The resonance frequency converges with 2nd order accuracy w.r.t the grid resolution Δx=Δy, as shown in Fig. 2, right. As reported before, the classical averaging techniques are only 1st order convergent and clearly not competitive. After symmetrization of the original material matrix (details will be explained in the full paper) the convergence is still 2nd order in this case. Although a general proof may be difficult, the accuracy will probably always outperform the classical schemes.

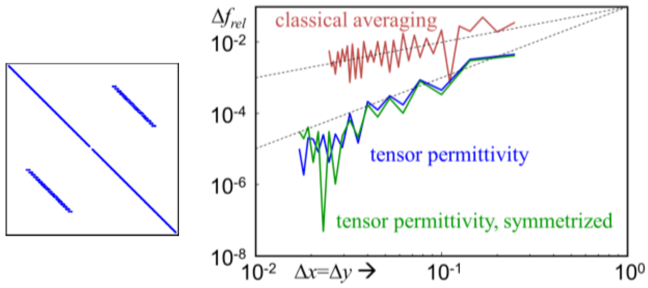

Fig. 2. Matrix pattern of $\mathbf{M}_\varepsilon$ and convergence of resonance frequency.

IV. Conclusion

The effective tensorial permittivity from the so-called subpixel smoothing has been implemented for FIT simulations with non-conformal dielectric interfaces. A symmetrization of the resulting material operator is necessary at least in time domain and can have an impact on the accuracy of the approach. Several symmetrization strategies have been implemented, going beyond what has been published before. The final presentation will give some more information here as well as a detailed convergence analysis for the tested variants.

References

[1] R. Schuhmann and T. Weiland, "FIT / FDTD Schemes on Non-Cartesian Computational Grids," *Proc. of the ACES Conference*, Monterey, CA, pp. 273-280, 2002.

[2] T. Weiland, "A discretization method for the solution of Maxwell's equations for six-component fields," *Electr. Comm.* 31, pp. 116, 1977.

[3] B. Krietenstein, R. Schuhmann, P. Thoma, and T. Weiland, "The perfect boundary approximation technique facing the big challenge of high precision field computation," *Proc. of the XIX International Linear Accelerator Conference (LINAC)*, Chicago, USA, pp. 860-862, 1998.

[4] S. Dey and R. Mittra, "A locally conformal finite-difference time-domain algorithm for modeling three-dimensional perfectly conducting objects," *IEEE MGWL* 7(9), pp. 273-275, 1997.

[5] A. Farjadpour et al., "Improving accuracy by subpixel smoothing in the finite-difference time domain", *Opt. Lett.* 31(20), pp. 2972-2974, 2006.

[6] D.M. Shyroki, "Modeling of Sloped Interfaces on A Yee Grid," *IEEE Trans. Ant. Prop.* 59(9), pp. 3290-3295, 2011.

[7] G.R. Werner and J.R. Cary, "A stable FDTD algorithm for non-diagonal, anisotropic dielectrics," *J. Comp. Phys.* 226, pp. 1085-1101, 2007.

[8] C.A. Bauer, G.R. Werner, and J.R. Cary, "A second-order 3D electromagnetics algorithm for curved interfaces between anisotropic dielectrics on a Yee mesh," *J. Comp. Phys.* 230, pp. 2060-2075, 2011.

[9] R. Schuhmann and T. Weiland, "Stability of the FDTD algorithm on nonorthogonal grids related to the spatial interpolation scheme," *IEEE Trans. Mag.* 34(5), pp. 2751-2754, 1998.

[10] H. Krüger, "*Zur numerischen Berechnung transienter elektromagnetischer Felder in gyrotropen Materialien*," PhD Thesis, Technische Universität Darmstadt, Germany, 2000.

[11] C. Classen, B. Bandlow, and R. Schuhmann, "Local approximation based material averaging approach in the Finite Integration Technique," *IEEE Trans. Magn.* 48(2), pp. 447-450, 2012.