Coupling of Different Dimensions in Finite Element Analysis for Solving Laplace and Poisson Equations

Tommi Peussa and Anouar Belahcen
Aalto University, Department of Electrical Engineering
P.O. Box 13000, FI-00076 AALTO, Finland
tommi.peussa@aalto.fi

***Abstract*—The coupling of different dimensionalities in finite element modeling is presented. 3D and 2D models as well as 2D and 1D models are coupled. The coupled models are used to solve Laplace and Poisson equation with Dirichlet boundary conditions. The results show that the coupled model give similar results to full 3D or 2D models but with significantly lower number of elements.**

I. Introduction

The Finite Element Method (FEM) is a standard procedure for solving various technical and physical problems. Even though the computing capacity has increased significantly during last years and decades, the FEM can still produce so many degrees of freedom (DOF) for large models that the solution of such models is unpractical. The needed computing capacity for large time-dependent 3D models is even today impractically large. The number of DOFs gets even larger with coupled problems, e.g. mechanical-electro-magnetical, [1],[2], and with non-linear problems. Also, for certain types of problems, e.g. eddy-current problems, the need for true 3D modeling is necessary.

The number of DOFs can be reduced simply by using lower dimensional models [3] or using e.g. slice models, which are series of 2D models, to model skewing in electrical machines [4].

The coupling of different dimensions has recently gained some interest, e.g. [5], [6]. In [5] the coupling is done stepwise: first a 2D solution is obtained for a part of the domain and then this solution is used as a boundary condition for 3D domain. In [6] the problem is divided into sequence of subproblems, some of lower dimensions, and then the complete solution is expressed as the sum of subproblem solutions.

This paper presents a preliminary method for coupling different dimensions directly into one FE-model. Here only simple problems are solved but the method is planned to be used in more complex problems as well. The aim of this paper is to show that it is possible to couple different dimensions into one FE-model.

II. Theory

The Laplace and Poisson equations have many physical interpretations. Here two different cases are presented.

The Laplace equation,

$$\nabla^2 u = 0, \tag{1}$$

defines the electric potential, u, in a capacitor with boundary conditions defining the potentials of the capacitor plates.

The Poisson equation,

$$-k\nabla^2 u = q, \tag{2}$$

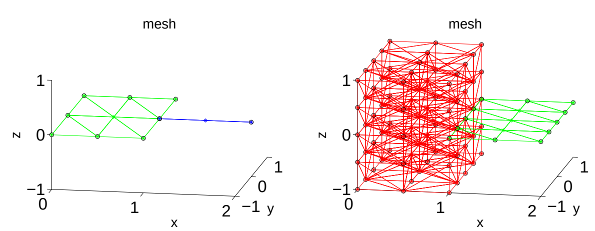
is the stationary heat equation for a volume that contains heat source, where u is the temperature, q is the heat source density and k is the thermal conductivity. If $k = 1$ and $q = 1$, the Poisson equation takes the form:

$$-\nabla^2 u = 1. \tag{3}$$

In FEM the equations may be transformed into weak form that can be approximated by set of linear equations. The resulting equation for Poisson equation is a matrix equation, $\mathbf{Su} = \mathbf{F}$. If the boundary condition is Dirichlet, the equations transforms into inner and boundary parts in matrix equation:

$$\begin{pmatrix} \mathbf{S}_{hh} & \mathbf{S}_{hb} \\ \mathbf{S}_{hb} & \mathbf{S}_{bb} \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ \mathbf{u}_b \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ \mathbf{f}_b \end{pmatrix}. \tag{4}$$

III. Model



(a) 2D-1D (b) 3D-2D

Fig. 1: FE mesh of 2D-1D model with 16 2D and 2 1D elements (a) and 3D-2D model with 192 3D and 16 2D elements (b).

The model is either 2D-1D model or 3D-2D model (see Fig. 1). The nodes in the domains are classified as inner nodes, $n_h^{(3D)}$ and $n_h^{(2D)}$ for 3D and 2D domains, respectively. Similarly there are boundary nodes, $n_b^{(3D)}$ and $n_b^{(2D)}$ for 3D and 2D domains, respectively. The common nodes, $n_c^{(3D)}$ and $n_c^{(2D)}$ are the nodes that are common for both domains. In the example in Fig. 1(b) the criteria would be $n_x = 1$, $-1 < n_y < 1$ and $-1 < n_z < 1$. The definitions are similar for 2D-1D case.

With these definitions, the matrix equation (4) for 3D part is

$$\begin{pmatrix} \mathbf{S}_{hh}^{(3D)} & \mathbf{S}_{hb}^{(3D)} & \mathbf{S}_{hc}^{(3D)} \\ \mathbf{S}_{hb}^{(3D)} & \mathbf{S}_{bb}^{(3D)} & 0 \\ \mathbf{S}_{hc}^{(3D)} & 0 & \mathbf{S}_{bc}^{(3D)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{h}^{(3D)} \\ \mathbf{u}_{b}^{(3D)} \\ \mathbf{u}_{c}^{(3D)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{h}^{(3D)} \\ \mathbf{f}_{b}^{(3D)} \\ \mathbf{f}_{c}^{(3D)} \end{pmatrix}, \tag{5}$$

and similarly for 2D part.

Coupling the nodes in the common boundary states that the nodes of lower dimension part are considered as nodes of the higher dimension part, $n_c^{(2D)} \subset n_c^{(3D)}$ and $n_c^{(1D)} \subset n_c^{(2D)}$. Thus in the common boundary, it is sufficient to refer only to solution in the higher dimension case, i.e. $\mathbf{u}_c^{(2D)} \rightarrow \mathbf{u}_c^{(3D)}$ and $\mathbf{u}_c^{(1D)} \rightarrow \mathbf{u}_c^{(2D)}$.

The coupling of the two domains is adjusted by a coupling parameter, c that is used for the coupling nodes in the common domain. The parameter is not necessary a single valued parameter but it depends on the type of the coupling. For "simple" coupling only the primary nodes are coupled, i.e. nodes that have exactly the same coordinates. For more "complex" coupling secondary nodes are coupled as well and then the value depends on the spatial location of the corresponding nodes. For simplicity in notation, the coupling part of the stiffness matrix is multiplied by single coupling parameter, c: $\mathbf{S}_c^{(2D)} \rightarrow c \cdot \mathbf{S}_c^{(2D)}$ or $\mathbf{S}_c^{(1D)} \rightarrow c \cdot \mathbf{S}_c^{(1D)}$.

With these definitions the matrix for the coupled domains for 3D-2D case is (and similarly for 2D-1D case)

$$\begin{pmatrix} \mathbf{S}_{hh}^{(3D)} & \mathbf{S}_{hb}^{(3D)} & \mathbf{S}_{hc}^{(3D)} + c\mathbf{S}_{hc}^{(2D)} & 0 & 0 \\ \mathbf{S}_{hb}^{(3D)} & \mathbf{S}_{bb}^{(3D)} & 0 & 0 & 0 \\ \mathbf{S}_{hc}^{(3D)} + c\mathbf{S}_{hc}^{(2D)} & 0 & \mathbf{S}_{bc}^{(3D)} + c\mathbf{S}_{bc}^{(2D)} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{S}_{hh}^{(2D)} & \mathbf{S}_{hb}^{(2D)} \\ 0 & 0 & 0 & \mathbf{S}_{hb}^{(2D)} & \mathbf{S}_{bb}^{(2D)} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \mathbf{u}_{h}^{(3D)} \\ \mathbf{u}_{b}^{(3D)} \\ \mathbf{u}_{c}^{(3D)} \\ \mathbf{u}_{h}^{(2D)} \\ \mathbf{u}_{b}^{(2D)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{h}^{(3D)} \\ \mathbf{f}_{b}^{(3D)} \\ \mathbf{f}_{c}^{(3D)} + \mathbf{f}_{c}^{(2D)} \\ \mathbf{f}_{h}^{(2D)} \\ \mathbf{f}_{b}^{(2D)} \end{pmatrix}. \tag{6}$$

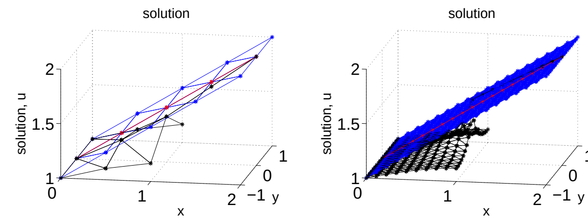
IV. Results

The Laplace equation is solved in simple case. Boundary conditions: $u(x) = 1$ on boundary $x = 0$ and $u(x) = 2$ on boundary $x = 1$. The mesh for the problem is shown in Fig. 1(a) and the result in Fig. 2.

The Poisson equation (3) is solved for simple case with zero Dirichlet boundary conditions. The source term is 1. The smaller case is the example mesh shown in Fig. 1(b) and the result is plotted in the Fig. 3(a). In Fig. 3(b) the number of elements is larger for the same problem.

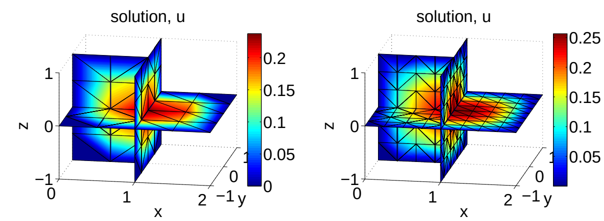
V. Conclusion

The coupling of different dimensions in FEA gives similar result as a full 3D or 2D model in simple cases like Poisson equation with Dirichlet boundary conditions. In coupled analysis the number of DOFs is significantly lower than in full 3D



(a) small model (b) larger model

Fig. 2: Solution of Laplace equation of 2D-1D model with 16 2D and 2 1D elements (a) and model with 324 2D and 9 1D elements (b). The red line is the 1D, the blue line is the 2D, and the black line is the 2D-1D solution.



(a) 208 elements (b) 1216 elements

Fig. 3: Solution of Poisson equation of 3D-2D model with 192 3D and 16 2D elements (a) and model with 1152 3D and 64 2D elements (b).

or 2D cases. In Table I the number of elements are compared in coupled and non-coupled analyses.

In the full paper example cases with different boundary conditions as well as a discussion on the choice of the coupling parameter will be presented.

TABLE I: NUMBER OF ELEMENTS IN COUPLED AND FULL ANALYSIS IN EXAMPLE CASES. CASES 1,2,3 AND 4 REFER TO FIGS. 2(a), 2(b), 3(a) AND 3(b), RESPECTIVELY.

case	coupled	full (3D or 2D)
case 1	18	32
case 2	333	648
case 3	208	384
case 4	1216	2304

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