# A Singularity Expansion Method for Analysis of Wireless Power Transfer System

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***Abstract*— Wireless power transfer via resonant inductive coupling has become a very topical research issue. However, there is no attempt to attribute such resonance to the eigenmode of the transmitter and receiver in regard to electromagnetic field. Although Singularity Expansion Method (SEM) was originally proposed to represent the transient electromagnetic response of electromagnetic configurations, it is extended to complex frequency domain in this paper to analyze the natural electromagnetic oscillations. To carry out efficient numerical implementation, SEM is casted in the form of moment method (MM), and the natural frequency of the resonant structure is found by means of the Muller's method. In terms of the resonant eigenmode field, the power transfer mechanism of resonant coils can then be clearly explained.**

***Index Terms*— Field pattern, moment method, singularity expansion method, wireless power transfer.**

## I. Introduction

Recently, an efficient wireless power transfer method, accomplished by means of magnetic resonance, was proposed [1]. The higher mid-range efficiency was claimed to depend on the quality factor of the resonant coils, whereas conventional electromagnetic induction is mainly dependent on the coupling coefficient between the transmitting and receiving coils.

The coupled-mode theory establishes a heuristic explanation of the mechanism of such power transfer system. Several equivalent circuit models have been accomplished to represent a resonant magnetic coupling system, because the coupled-mode theory is too complicated to synthesize the system. However, these approaches cannot distinguish the resonant-based method from induction method, and hence imposing unnecessary restriction on design improvement of the transmitter and receiver. Moreover, the parameters of equivalent circuit are usually abstracted by means of computational electromagnetics. The most satisfying method is to find the resonance directly using full-wave numerical technique, removing the assumption of lumped inductance and capacitance, which are in fact based on quasi-static approximation. Although finite element method (FEM) is a popular method to find resonance, in the regime of inductive resonance, i.e., an open-boundary eigenvalue problem, it requires heavy computational burden, even after the introduction of truncation techniques such as absorbing boundary and radiation boundary. Furthermore, while discussing the interaction between resonant transmitter and receiver, FEM suffers from serious numerical dispersion. As an integral equation based method, moment method (MM) is prevalent to solve open boundary problems, and is spared from the numerical dispersion. Therefore, the resonance or eigenvalue problem formulated using MM would be the most satisfying method to analyze magnetic resonance-based wireless power transfer systems.

In this paper, taking the advantage of singularity expansion method (SEM), the matrix formed by MM is used to find the resonant frequency and its corresponding eigenmode. A numerical approach of finding the poles (singularity), its corresponding mode field and coupling coefficient is presented. In the end, some concepts borrowed from the antenna community are used to augment the analysis of the resonance and electromagnetic interaction of a helical resonant structure.

## II. Formulation of Singularity Expansion Method

Considering a body with finite dimensions in free space and letting this body have a volume $V$ enclosed by a surface $S$, one can express the electric field $\mathbf{E}$ in terms of the current density as

$$\mathbf{E} = \int_V \mathbf{G}(\mathbf{r},\mathbf{r}',\omega)\cdot \mathbf{J}dV\,,\quad \mathbf{r},\mathbf{r}'\in V\,, \tag{1}$$

where $\mathbf{G}$ is the dyadic Green's function. Using the MM [2], the unknown current $\mathbf{J}$ can be expanded by basis functions with unknown $I_n$, and the approximation of above integral equation results in a matrix equation as follows

$$V_n = Z_{nm}(\omega) I_n\,,\quad n,m=1,\cdots,N\,, \tag{2}$$

where $Z_{nm}(\omega)$ is the discrete form ($N\times N$ matrix) of the kernel of the integral equation; and $V_n$ stands for the excitation function. Observing the typical transient responses of various complicated scatters, they appear to be dominated by a few sinusoids. Since these waveforms correspond to the poles in the frequency-domain, scatter is expected to produce large responses at frequencies near those poles, i.e., the natural frequencies. The self-resonant coils presenting in wireless power transfer, in the view of resonators, can be regarded as open resonators. Note that the Green's function $\mathbf{G}(\boldsymbol{r},\boldsymbol{r}',\omega)$ is an analytical function of $\omega$ except for $\omega = 0$. After zoning procedure, the resulting $Z_{nm}(\omega)$ must be entire functions except for local poles at $\omega = 0$. Based on the above argument, the solution of (2) can be expressed as [3]

$$I_n(\omega) = \frac{1}{\left|Z_{nm}(\omega)\right|} g_{nm}(\omega) V_n\,, \tag{3}$$

$$g_{nm}(\omega) = (-1)^{n+m}\left|Z_{nm}(\omega)\right|'\,,$$

where $Z_{nm}(\omega)'$ is the minor of the element $z_{ij}$ of matrix $Z_{nm}(\omega)$. Since the determinant is a sum of products of the matrix elements, there are only pole singularities in the finite complex-frequency plane. Consequently, by expanding $Z_{nm}(\omega)$

in power series, the currents induced on finite-size perfect conducting objects in free space is formulated as

$$\mathbf{J}(\mathbf{r},\omega)=\sum_{i=1}^{m}\eta_i(\omega)\,\mathbf{j}_i(\mathbf{r})/(\omega-\omega_i)^m+\varphi(\mathbf{r},\omega). \quad (4)$$

Except the residual $\varphi$ (**r**,ω), which can be ignored when one discusses the resonance at $\omega_i$, the main task of SEM is to find the coupling coefficient $\eta_i(\omega)$ and resonant mode $\mathbf{j}_i(\mathbf{r})$.

## III. Implementation of Singularity Expansion Method

To determine the resonant mode and its associated coupling coefficient, the first step is to find the resonant frequencies for a given geometry, which is necessary to solve

$$|Z_{nm}(\omega)|=0. \quad (5)$$

This is the problem of finding the roots of nonlinear equation. Newton's method is a commonly used routine. To avoid applying $d|Z_{nm}(\omega)|/d\omega$ in the calculation, Muller's method is employed [4]. In this method, the initial estimate of the poles is not needed and multiple roots can be found.

In Muller's method, three points ($\omega_{i-1}$, $|Z_{nm}(\omega_{i-1})|$), ($\omega_i$, $|Z_{nm}(\omega_i)|$), ($\omega_{i+1}$, $|Z_{nm}(\omega_{i+1})|$) are used to approximate the local behavior of $|Z_{nm}(\omega)|$. Such approximation is to construct the parabola that goes through all these points. The parabola is unique. By defining $|Z_{nm}(\omega_i)| = f(\omega_i)$, it can be formulated as:

$$\begin{aligned} p(\omega) &= f(\omega_i)+f(\omega_i,\omega_{i-1})(\omega-\omega_i) \\ &\quad + f(\omega_i,\omega_{i-1},\omega_{i-2})(\omega-\omega_i)(\omega-\omega_{i-1}) \\ &= a_0+a_1\omega+a_2\omega^2, \end{aligned} \quad (6)$$

where

$$f(\omega_i,\omega_{i-1})=\frac{f(\omega_i)-f(\omega_{i-1})}{\omega_i-\omega_{i-1}}, \quad (7)$$

$$f(\omega_i,\omega_{i-1},\omega_{i-2})=\frac{f(\omega_i,\omega_{i-1})-f(\omega_{i-1},\omega_{i-2})}{\omega_i-\omega_{i-2}}, \quad (8)$$

It is obvious that the values of $a_i$ in (6) are all constants, by which the root of this parabola can be obtained

$$\omega=\frac{2a_0}{-a_1\pm(a_1^2-4a_0a_2)^{1/2}}. \quad (9)$$

The larger root will take the place of $\omega_{i+1}$ to repeat the above process until some criteria are met. To find the next root of the determinant, the available pole is removed by

$$|Z_{nm}(\omega)|/(\omega^2-\omega_0^2) \quad (10)$$

After substituting the natural frequency $\omega_0$ into $Z_{nm}(\omega)$, the natural mode $J_n$ and coupling coefficients $H_n$ can be obtained by solving the following matrix equations,

$$H_n^T Z_{nm}=0, \quad (11)$$

$$Z_{nm}I_n=0. \quad (12)$$

## IV. Numerical Examples

Taking the spiral coil in [1], for instance, the self-resonance of the spiral structure is investigated. Therefore, the feed loop in the original paper can be omitted. This is a purely metallic thin-wire structure, which can be formulated by electric field integral equation and solved by the method proposed in above sections. At resonant frequency, i.e. 10.3 MHz, its current distribution is shown in Fig. 1. Such current distribution profile and resonant frequency agree fairly well with the reported experimental results.

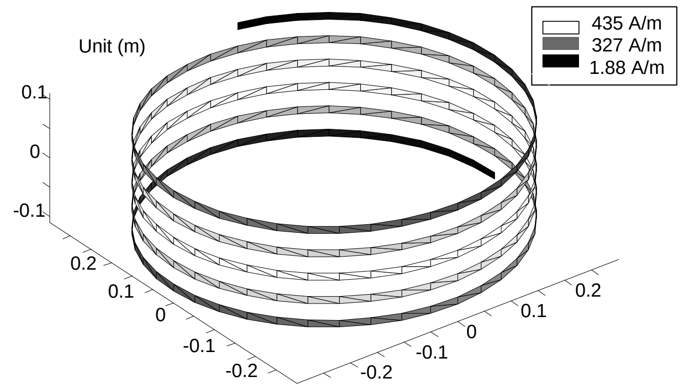


Fig. 1. Current distribution of the spiral coil proposed in [1] at resonance.

By plotting the near-field pattern of the isolated and coupled spirals at resonance, as shown in Fig. 2, it can be seen that the single spiral shows a nearly omni-directional pattern accompanied by intense radiation, while the coupling of the receiving spiral produces a directional near-field pattern with much weaker radiation. In the view of an antenna, a traveling wave is formed by these two coupling spirals, which works just like Yagi-Uda antenna arrays, but most energy is collected by the impedance-matched receiver.

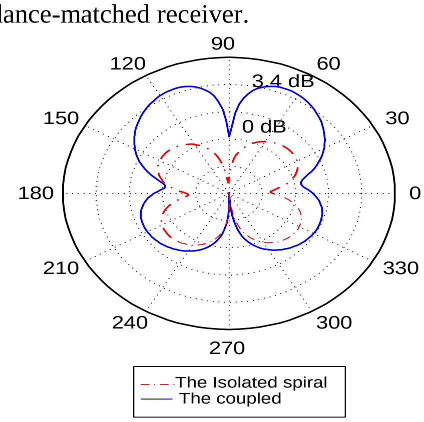


Fig. 2. Near field pattern of the isolated and the coupled resonant spiral

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