Circuit-Oriented Solution of Drude Dispersion Relation by the FD²TD

Silvano Cruciani[1], Valerio De Santis[2], Francesca Maradei[3] and Mauro Feliziani[1]
[1]Department of Industrial and Information Engineering and Economics, University of L'Aquila, L'Aquila, Italy
[2]IT'IS Foundation, Zurich, Switzerland
[3]Department of Electrical Engineering, Sapienza University of Rome, Rome, Italy
silvano.cruciani@univaq.it , desantis_valerio@msn.com , francesca.maradei@uniroma1.it , mauro.feliziani@univaq.it

***Abstract*— This paper deals with the time-domain numerical calculation of electromagnetic (EM) fields in media described by linear Drude dispersive models. The frequency-dependent finite-difference time-domain (FD²TD) method is applied to solve multipole Drude equations using two equivalent circuit models.**

***Index Terms*— Finite difference methods, Time domain analysis, Computational electromagnetics, Equivalent circuit, Propagation losses.**

I. Introduction

The finite-difference time-domain (FDTD) method has been used to solve problems of electromagnetic wave propagation in dispersive media since many years [1]-[3]. In the recent past an improved formulation of the frequency-dependent finite-difference time-domain (FD²TD) method has been presented to analyze media characterized by multipole Debye dispersion relation [4]. In the present work, the FD²TD method proposed in [4] is extended to model media with multipole Drude linear dispersion using circuit methods [5]-[6] developed for other kinds of applications [7]-[8]. For the solution of multipole Drude media, two different circuit methods, named as CIRC and PCRC, are proposed.

II. Mathematical Formulation

Electromagnetic fields in the Laplace domain are described by Maxwell's curl equations:

$$\nabla\times\boldsymbol{E}(s) \;=\; -s\mu\boldsymbol{H}(s) \tag{1a}$$

$$\nabla\times\boldsymbol{H}(s) \;= s\varepsilon_0\hat{\varepsilon}_r(s)\boldsymbol{E}(s) \tag{1b}$$

where $\boldsymbol{E}(s)$ and $\boldsymbol{H}(s)$ are the electric and magnetic fields, respectively, s is the Laplace variable, μ the permeability, ε_0 the free space permittivity, and $\hat{\varepsilon}_r$ the frequency-dependent complex relative permittivity. For a Drude linear dispersive medium, $\hat{\varepsilon}_r(s)$ can be written as $\hat{\varepsilon}_r(s)=\varepsilon_\infty+\chi(s)+\sigma/(s\varepsilon_0)$, being ε_∞ the infinite relative permittivity, $\chi(s)$ the dielectric susceptibility and σ the conductivity. In case of a multipole Drude medium, $\chi(s)$ is given by:

$$\chi(s) \;=\sum_{m=1}^{M}\chi_m(s) \;=\sum_{m=1}^{M}\frac{\omega_m^{\;2}}{s\left(s+v_{c,m}\right)} \tag{2}$$

where $v_{c,m}$ is the Drude pole, ω_m is the inverse of the relaxation time and M is the number of poles [1]. The admittance $Y_m(s)$ of the generic mth pole is given by:

$$Y_m(s)=s\varepsilon_0\chi_m(s)=\varepsilon_0\,\omega_m^{\;2}\big/\left(s+v_{c,m}\right) \tag{3}$$

which can be rewritten in a more suitable form for circuit-oriented applications as:

$$Y_m(s)=1\big/\left(R_m+sL_m\right) \tag{4}$$

where the mth inductance and resistance are given by:

$$L_m=1/\left(\varepsilon_0\omega_m^{\;2}\right) \tag{5a}$$

$$R_m=v_{c,m}/\left(\varepsilon_0\omega_m^{\;2}\right) \tag{5b}$$

For a multipole Drude dispersion media, the constitutive relation between the electric field $\boldsymbol{E}$ and the total dispersion current $\boldsymbol{J}$ is given by $\boldsymbol{J}=s\varepsilon_0\hat{\varepsilon}_r(s)\boldsymbol{E}$ and can be represented by the equivalent circuit shown in Fig.1 when assuming $G_0=\sigma$ and $C_\infty=\varepsilon_0\varepsilon_\infty$. This circuit is very similar to that proposed in [7] to analyze impedance boundary conditions. It means that the same methods used in [7] can be applied for the solution of multipole Drude dispersion media by the FD²TD method as described in the following.

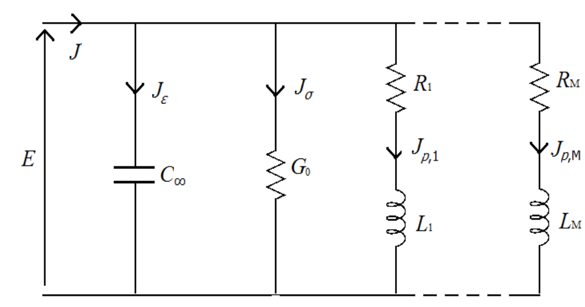

Fig. 1. Equivalent circuit of a multipole Drude dispersion model.

Transforming (1) into the time domain and applying the FD²TD method in one-dimension (1D) for the sake of simplicity and without any loss of relevance, it yields:

$$\frac{E_{i+1}^{n}-E_{i}^{n}}{\Delta x}=-\mu\frac{H_{i+1/2}^{n+1/2}-H_{i+1/2}^{n-1/2}}{\Delta t} \tag{6a}$$

$$\frac{H_{i+1/2}^{n+1/2}-H_{i-1/2}^{n+1/2}}{\Delta x}=G_0\frac{E_i^{n+1}+E_i^{n}}{2}+C_\infty\frac{E_i^{n+1}-E_i^{n}}{\Delta t}+\sum_{m=1}^{M}J_{p,m,i}^{n+1/2} \tag{6b}$$

where the superscript n and the subscript i refer respectively to the time iteration ($t=n\Delta t$) and spatial iteration ($x=i\Delta x$) being Δt the time step and Δx the spatial discretization, while $J_{p,m}$ is the mth polarization current that can be obtained in two different ways [7]. In the first method named CIRC, $J_{p,m}$ is directly calculated in time domain applying the finite difference scheme to the constitutive relationships of the $R_m L_m$ series circuit branch yielding to:

$$J_{p,m}^{n+1/2} = \left(L_m/\Delta t + R_m/2\right)^{-1}\left[\left(L_m/\Delta t - R_m/2\right)J_{p,m}^{n-1/2} + E^n\right] \quad (7)$$

$J_{p,m}$ can be also obtained using a piecewise constant recursive convolution (PCRC) method as:

$$J_{p,m}(t) = \int_{-\infty}^{t} y_m(t-\tau)E(\tau)d\tau \quad (8)$$

where the transient admittance $y_m(t)$ is analytically given by:

$$y_m\left(t\right) = \Im^{-1}\left[Y_m\left(s\right)\right] = e^{-\frac{R_m}{L_m}t}/L_m \quad (9)$$

being $\Im^{-1}$ the inverse Laplace transform. $J_{p,m}$ can be then calculated at discrete time $t=(n+1)\Delta t$ as:

$$J_{p,m}^{n+1/2} = \frac{1}{L_m}\int_0^{(n+1/2)\Delta t} e^{-\frac{R_m}{L_m}\left[(n+1/2)\Delta t-\tau\right]} E\left(\tau\right)\, d\tau \quad (10)$$

and applying the PCRC integration scheme it yields:

$$J_{p,m}^{n+1/2} = e^{-\frac{R_m}{L_m}\Delta t} J_{p,m}^{n-1/2} + \left(1 - e^{-\frac{R_m}{L_m}\Delta t}\right)\frac{E^n}{R_m} \quad (11)$$

After calculation of $J_{p,m}$, the FD^2TD solution of system (6) does not present any difficulty.

III. Numerical Results

Consider a FD^2TD simulation of a TEM propagation inside a dispersive 2-pole Drude material slab of thickness d = 30 mm characterized by $\varepsilon_\infty=1$, $\omega_1=2\pi\ 12\cdot10^9$ s^{-1}, $\omega_2=2\pi\ 30\cdot10^9$ s^{-1}, $\nu_{c,1}=7\cdot10^9$ s^{-1}, $\nu_{c,2}=25\cdot10^9$ s^{-1} and $\sigma=0.05$ S/m and surrounded by free space. The waveform of the electric field excitation is a Gaussian pulse described by $E^i(t)=\exp(-(wt-4)^2)$ V/m with $w=1.6\cdot10^{11}$ s^{-1}. The 1D computational domain is discretized by 1500 cells assuming Δx=37.5 μm and the slab is located between cells N_1 = 401 and N_2 = 1200. The time step is Δt=0.125 ps. Absorbing boundary conditions (ABC) are applied to the extremity boundaries of the computational domain in free space to avoid any field reflection. For the considered configuration the analytical solution is available via the inverse Fourier transform. The examined configuration and the comparison between the results of the proposed circuit-oriented FD^2TD simulations and the analytical solution at $t=1500\Delta t$ are reported in Fig. 2.

In order to evaluate the precision of the numerical results the frequency-dependent wave number $k_{FDTD}=-log(E_{FFT}(\omega,P_2)/E_{FFT}(\omega,P_1))/(j(P_2-P_1))$ is calculated by the FDTD methods via the Fast Fourier Transform (FFT) assuming Δx=37.5 μm and Δt=0.125 ps. $P_1 = (N_1 + 10)\Delta x$ and $P_2 = P_1 + 10\Delta x$ are two points inside a single pole Drude material characterized by ε_∞ = 1.5, $\omega_1 = 2\pi\ 28.7\cdot10^9$ s^{-1}, $\nu_{c,1} = 20\cdot10^9$ s^{-1} and σ = 0 S/m [2]. The computation is stopped at $t=1500\Delta t$ and k_{FDTD} for the CIRC and PCRC methods is compared with the exact value $k_{exact}=\omega(\mu_0\varepsilon_0\,\hat{\varepsilon}_r(\omega))^{1/2}$ as shown in Fig. 3 revealing a very good agreement.

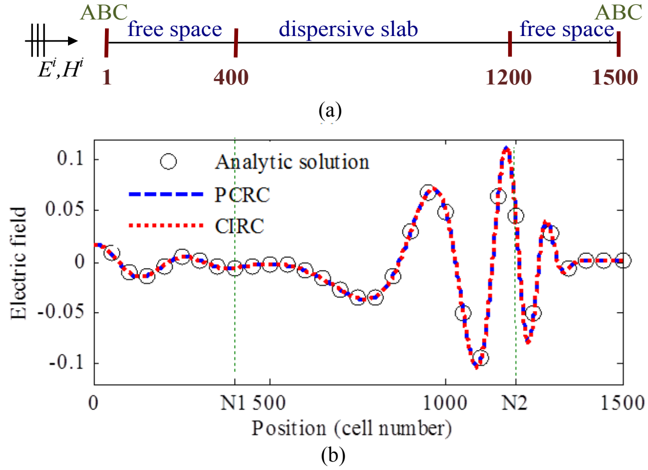

Fig. 2. Computational domain (a). Comparison between analytic and FDTD computed solutions of electric field at t=1500 Δt for the Drude media (b).

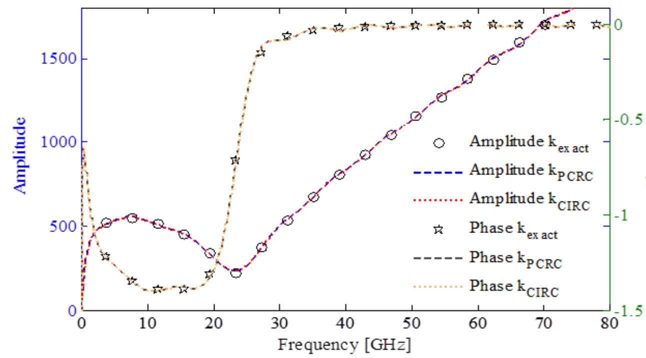

Fig. 3. Frequency-dependent wave number k for the Drude media: comparison between exact value and FFT of the FDTD computed solution.

References

[1] A. Taflove and S.C. Hagness, *Computational Electrodynamics: The Finite Difference Time Domain, 3rd Ed.*, Norwood, MA, Artech House, 2005.

[2] K. Kunz and R. Luebbers, *The Finite Difference Time Domain Method for Electromagnetics*, Boca Raton, FL: CRC Press, 1993.

[3] J. Young, R. Nelson, "A summary and systematic analysis of FDTD algorithms for linearly dispersive media," *IEEE Antennas Propag. Mag.*, vol. 43, no. 1, pp. 61-126, Feb. 2001.

[4] M. Feliziani, S. Cruciani, V. De Santis and F. Maradei, "FD2TD analysis of electromagnetic field propagation in multipole debye media with and without convolution," *Progress In Electromagnetics Research B*, vol. 42, pp. 181-205, 2012.

[5] M. Feliziani and F. Maradei, "FEM solution of time-harmonic electromagnetic fields by an equivalent electrical network," *IEEE Trans. Magnetics*, vol. 36, no. 4, pp. 938–941, July 2000.

[6] M. Feliziani and F. Maradei, "Circuit-Oriented FEM: solution of circuit–field coupled problems by circuit equations," *IEEE Trans. Magn.*, vol. 38, no. 2, pp. 965-968, March 2002.

[7] M. Feliziani and S. Cruciani, "FDTD modeling of Impedance Boundary Conditions by equivalent LTI circuits," *IEEE Trans. Microw. Th. Techn*, vol. 60, no. 12, pp. 3656 – 3666, Dec. 2012.

[8] M. Feliziani, "Subcell FDTD modeling of field penetration through lossy shields," *IEEE Trans. Electromag. Compat.*, vol. 54, no. 2, pp. 299-307, Apr. 2012.