Isogeometric Finite Elements with Surface Impedance Boundary Conditions

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***Abstract*—A high order surface impedance boundary condition is introduced in a two-dimensional variational formulation in terms of the magnetic vector potential. The problem is discretized with isogeometric finite elements, a recent method that uses NURBS functions both to describe the geometry and the solution field. The method has the advantage that it allows an exact computation of the curvature of the geometry, which is required by high order surface impedance boundary conditions. The proposed method is applied to the computation of per-unit-length parameters of multiconductor transmission lines.**

***Index Terms*—Eddy currents, Finite element methods, Transmission lines.**

I. Introduction

Isogeometric Analysis (IGA) [1] is a discretization technique which was introduced to simplify the interaction between Computer Aided Design (CAD) software and numerical solvers. The discretization in IGA is done with Non-Uniform Rational B-Splines (NURBS), a family of functions that is widely used in CAD [2]. The main advantage of IGA with respect to standard Finite Element Methods (FEM) is that it maintains the representation of the geometry given by CAD. This is particularly interesting for the implementation of high-order surface impedance boundary conditions (SIBCs), because the curvature can be computed exactly at any point. The use of NURBS together with SIBCs was already applied in [3], in that case using the boundary element method (BEM) to discretize an integral equation for the computation of per-unit-length (p.u.l.) parameters of multiconductor transmission lines. In this work we present the variational formulation of the same problem, and we solve it numerically with IGA.

The same problem was already discretized with FEM in [4]. With respect to that work, our formulation i) applies the SIBC also to the conductors where the current is imposed, ii) includes high-order SIBCs, iii) accurately takes into account the curvature with the use of NURBS.

II. Magnetic Vector Potential Formulation

Consider a set of N infinitely long parallel conductors, with cross sections Ω_i, $i = 1, \ldots, N$. Each conductor is assumed to have electrical conductivity σ_i, permittivity ε_i, and magnetic permeability equal to the permeability of free space, $\mu_i = \mu_0$.

We consider the time-harmonic eddy-current model, with a two-dimensional formulation based on the magnetic vector potential $\mathbf{A} = A\mathbf{e}_z$, as in [5]. Splitting the potential into "source" and "eddy" components, $A = A^s + A^e$, it can be seen that A^s is constant in each conductor. Then, being $\operatorname{div}\mathbf{A} = 0$ in 2D, the eddy component A^e satisfies the following equation in each conductor

$$\nabla^2 A^e_{\text{int}} = \mathrm{j}\omega\mu_0\sigma_i A^e_{\text{int}} \qquad \text{in } \Omega_i, \tag{1}$$

and in the nonconducting domain

$$\nabla^2 A^e_{\text{ext}} = 0 \qquad \text{in } \Omega_0. \tag{2}$$

Denoting by Γ_i the boundary of (the cross section of) each conductor, and by $\mathbf{n}$ the unit normal vector exterior to Ω_0, the equations for A^e are completed with interface conditions on Γ_i:

$$[A^e]_{\Gamma_i} = -A^s, \ \left[\frac{\partial A^e}{\partial \mathbf{n}}\right]_{\Gamma_i} = 0, \tag{3}$$

where the brackets denote the jump on the interface, $[A^e] = A^e_{\text{int}} - A^e_{\text{ext}}$. The problem is completed with a radiation condition that, in practice, is replaced by truncating the infinite domain at the boundary $\Gamma_R = \{(x, y) : x^2 + y^2 = R^2\}$, and applying the absorbing boundary condition [6, Sect. 4.6]

$$\frac{\partial A^e_{\text{ext}}}{\partial \mathbf{n}} + \frac{A^e_{\text{ext}}}{2R} = 0. \tag{4}$$

Since the source component A^s is unknown, a condition on the intensity flowing in each conductor is also needed

$$\int_{\Gamma_i} \frac{1}{\mu_0} \frac{\partial A^e_{\text{ext}}}{\partial \mathbf{n}} = I_i, \quad i = 1, \ldots, N. \tag{5}$$

III. Surface Impedance Boundary Conditions

The general idea of applying SIBCs is to replace the solution of the problem inside the conductor given by (1), with an approximate boundary condition that replaces the field A^e_{int}. The method is valid under the condition of skin effect, that is, in each conductor the penetration depth $\delta = \sqrt{2/\omega\mu_0\sigma}$ is much smaller than the characteristic size of the conductor cross section.

Defining $\alpha = \sqrt{2\mathrm{j}}$, and using the interface conditions (3), the first order (Leontovich) and second order (Mitzner) SIBCs are [5]

$$\frac{\partial A^e_{\text{ext}}}{\partial \mathbf{n}} = -Z_L(A^e_{\text{ext}} - A^s) = \frac{-\alpha}{\delta}(A^e_{\text{ext}} - A^s), \tag{6}$$

$$\frac{\partial A^e_{\text{ext}}}{\partial \mathbf{n}} = -Z_M(A^e_{\text{ext}} - A^s) = \frac{-2\alpha^2}{\delta^2 c + 2\alpha\delta}(A^e_{\text{ext}} - A^s), \tag{7}$$

where $c = c(\xi)$ is the (signed) curvature of the contour of the cross section.

IV. Weak Formulation

The weak formulation of the problem is obtained from (2), applying the absorbing boundary condition (4) and the SIBCs (6) or (7):

$$\int_{\Omega_0} \nabla A^e_{\text{ext}} \cdot \nabla \overline{v} + \int_{\Gamma_R} \frac{A^e_{\text{ext}}}{2R}\overline{v} + \sum_{i=1}^{N} \int_{\Gamma_i} Z_{(L,M)}(A^e_{\text{ext}} - A^s)\overline{v} = 0. \tag{8}$$

The equations must be completed with the intensity condition (5), that is also modified by applying the corresponding SIBC, either (6) or (7).

V. NURBS and Isogeometric Analysis

From an ordered knot vector $\Xi = \{0 = \zeta_1, \ldots, \zeta_{n+p+1} = 1\}$, n univariate B-spline basis functions of degree p are computed using the Cox-De Boor formula:

$$\mathsf{B}_{k,0}(\zeta) = \begin{cases} 1, & \text{if } \zeta_k \le \zeta < \zeta_{k+1} \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

$$\mathsf{B}_{k,p}(\zeta) = \frac{\zeta - \zeta_k}{\zeta_{k+p} - \zeta_k}\mathsf{B}_{k,p-1}(\zeta) + \frac{\zeta_{k+p+1} - \zeta}{\zeta_{k+p+1} - \zeta_{k+1}}\mathsf{B}_{k+1,p-1}(\zeta). \tag{10}$$

Bivariate B-splines are simply defined from the previous formula using tensor products. Bivariate NURBS basis functions of degree (p, q) are defined as rational B-splines by associating a positive weight to each B-spline function.

$$\hat{N}_{k_1,k_2}(\boldsymbol{\zeta}) = \frac{w_{k_1,k_2} B_{k_1,p}(\zeta_1), B_{k_2,q}(\zeta_2)}{\sum_{j_1=1}^{n}\sum_{j_2=1}^{m} w_{j_1,j_2} B_{j_1,p}(\zeta_1), B_{j_2,q}(\zeta_2)} \tag{11}$$

defined for $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ in the unit square. A NURBS surface can be constructed by associating a control point to each basis function, to give a parametrization of the form (see [2])

$$\mathbf{F}(\boldsymbol{\zeta}) := \sum_{k_1=1}^{n}\sum_{k_2=1}^{m} \hat{N}_{k_1,k_2}(\boldsymbol{\zeta})\mathbf{C}_{k_1,k_2}. \tag{12}$$

For the discretization of (8) we use IGA, a numerical technique which is based on NURBS. Assuming that the domain Ω_0 is given as a NURBS geometry, like in (12), and invoking the isoparametric concept, the same space of NURBS functions is used for the discretization of the field in (8), which takes the form

$$A^e_{\text{ext}} = \sum_{k_1,k_2=1}^{n,m} A_{k_1,k_2} N_{k_1,k_2}, \quad \text{with } N_{k_1,k_2} = \hat{N}_{k_1,k_2} \circ \mathbf{F}^{-1}. \tag{13}$$

The main advantage of IGA is that the CAD geometry can be exactly preserved, without generating a mesh. Moreover, the computed solution is more smooth than the one given by FEM, since NURBS basis functions of degree p have up to $p-1$ continuous derivatives. See [1] for more details on IGA.

VI. Numerical Results

The implementation has been first validated by solving the canonical case of two parallel circular copper conductors, with conductivity $\sigma = 5.8 \times 10^7$ S/m. The diameter of each conductor is 2 mm, and the distance between their centers is 4 mm. With the numerical solution we have computed the p.u.l. resistance, following the procedure described in [5] and comparing numerical results with the analytical solution [7]. The relative errors, shown in Fig. 1, are in good agreement with those obtained in [3] and [5] using BEM.

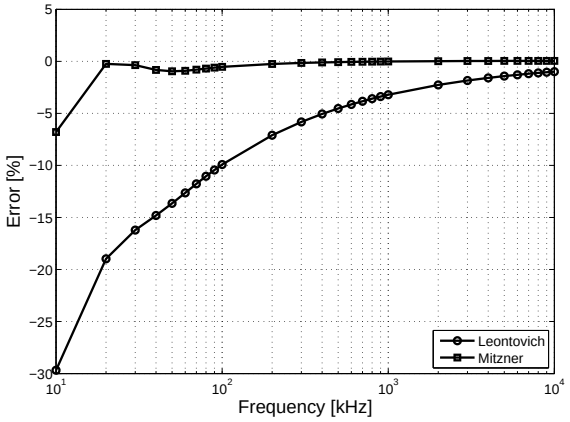

Figure 1. Relative error in p.u.l. resistance for two circular copper cables of diameter 2 mm. Distance between the centers of the conductors is 4 mm.

VII. Conclusion

In the full paper we will give more details on the implementation and on the computational cost of the above IGA formulation. We will apply the method to other cable geometries, as the ones already presented in [3] and [5], in order to carry out a deep error analysis by comparison with the already validated BEM formulation. Furthermore, the advantages of each formulation will be outlined.

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