Asymptotic boundary element methods for thin conducting sheets in two dimensions

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***Abstract*—Shielding sheets are commonly used in the protection of electronic devices. With their large aspect ratios they become a serious issue for the direct application of the finite element method, as many small cells are required to resolve the sheets, as well as the direct application of the boundary element method due to the occuring almost singular integrals.**

Impedance transmission conditions (ITCs) allow for finite element formulations in the exterior of the sheet mid-line or boundary element formulations on this mid-line only. We propose and analyse boundary element methods of second kind for ITCs of two different types for the time-harmonic eddy current problem in two dimensions.

***Index Terms*—Computational electromagnetics, Eddy currents, Electromagnetic shielding, Integral equations.**

I. Introduction

The time-harmonic eddy current model [5] (time convention $\exp(-i\omega t)$, $\omega > 0$) in two dimensions reads

$$\mathbf{curl}_{2D}\, e(\mathbf{x}) = \mathrm{i}\omega\mu(\mathbf{x})\mathbf{h}(\mathbf{x}), \tag{1}$$

$$\mathrm{curl}_{2D}\, \mathbf{h}(\mathbf{x}) = \sigma(\mathbf{x})e(\mathbf{x}) + j_0(\mathbf{x}), \tag{2}$$

where the 2D rotation operators are defined by $\mathbf{curl}_{2D} = (\partial_y, -\partial_x)^\top$ and $\mathrm{curl}_{2D} = (-\partial_y, \partial_x)$, e and $\mathbf{h}$ denote the out-of-plane electric and in-plane magnetic fields, σ is the (constant) conductivity of the thin sheet of thickness d, and we assume it to vanish elsewhere for simplicity, μ is the permeability (also constant inside the sheet), and j_0 is the out-of-plane imposed current whose location is well separated from the conductor. Inserting (1) in (2) outside the thin conductor and using the identiy $\mathrm{curl}_{2D}\,\mathbf{curl}_{2D} = -\Delta$ we obtain

$$-\Delta\, e = \mathrm{i}\omega\mu_0 j_0, \tag{3}$$

where we assume non-magnetic material outside the sheet, *e. g.*, air. Furthermore, let $\gamma = \sqrt{-i\omega\mu\sigma} = (-1 + \mathrm{i})\sqrt{\omega\mu\sigma/2}$.

II. Impedance transmission conditions

Several impedance transmission conditions have been derived for the time-harmonic eddy-current model, so the shielding element [1], the thin sheet conditions [2]–[3] and recently by asymptotic expansions [6]–[9]. Using those ITCs we can state the differential equation (3) up to the mid-line Γ of the thin sheet, where two conditions couple electric or magnetic fields on both sides. To define those we introduce the notation for the mean and jump of both sides

$$[v] = v^+ - v^-, \qquad \{v\} = \tfrac{1}{2}\left(v^+ + v^-\right), \tag{4}$$

where v stands either for the electric field e or the tangential component of the magnetic field $\mathbf{h}\cdot\mathbf{n}^\perp = \partial_n e/(\mathrm{i}\omega\mu)$, where $\mathbf{n}$ is the normalised normal vector as shown in Fig. 1 and $\mathbf{n}^\perp = (n_2, -n_1)^\top$ the normalised tangential vector. Note, that the above mentioned ITCs are derived for smooth sheets without kinks or endings, *e. g.*, by neglecting derivatives along the sheet. In this paper we consider two types of ITCs.

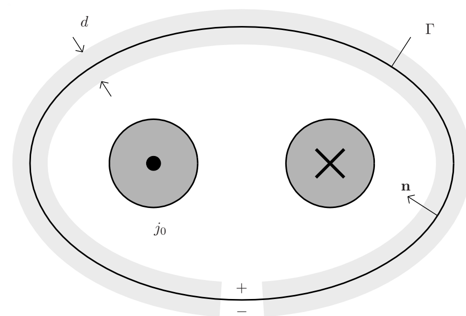

Figure 1: Limit geometry for $d \to 0$ for the impedance conditions and integral equations. The original sheet of thickness d is indicated with light shading.

A. Impedance transmission conditions of type I

For the most simple impedance transmission conditions the electric field is continuous over Γ and the jump of the magnetic field is proportional to the electric field. Using (1) we can write directly for the (mean of the) electric field

$$[\nabla e\cdot\mathbf{n}] - \beta_1\{e\} = 0, \tag{5a}$$

$$[e] = 0. \tag{5b}$$

Impedance transmission conditions of type I are for example the so called ITC-1-0 and ITC-1-1, both derived by asymptotic expansions, see [7]-[9], where the only parameter for the two models is given by

$$\beta_1^{\text{ITC-1-0}} = \gamma^2 d, \qquad \beta_1^{\text{ITC-1-1}} = \gamma^2 d\left(1 + \tfrac{1}{6}\gamma^2 d^2\right). \tag{6}$$

The shielding element [1] can be seen as an ITC similar to (5) with an additional term with $\partial_\Gamma^2\{e\}$ arising in (5a), see [9].

B. Impedance transmission conditions of type II

Currently, the thin sheet conditions [3] which are based on a similar idea as [2] are best known. They have the form

$$[\nabla e \cdot \mathbf{n}] - \beta_1 \{e\} \qquad = 0, \tag{7a}$$

$$[e] \qquad - \beta_2 \{\nabla e \cdot \mathbf{n}\} = 0, \tag{7b}$$

where

$$\beta_1^{\mathrm{MB}} = 2\gamma \tanh(\gamma \tfrac{d}{2}), \qquad \beta_2^{\mathrm{MB}} = \tfrac{2}{\gamma} \tanh(\gamma \tfrac{d}{2}). \tag{8}$$

In [9] it has been shown that these conditions even with their more complex structure does not improve the accuracy to the simple ITC-1-0 or the shielding element [1]. An improvement for both large and small skin-depth to sheet thickness ratios can be observed by chosing [9]

$$\begin{aligned} \beta_1^{\text{ITC-2-1}} &= \frac{2\gamma \sinh\left(\gamma \frac{d}{2}\right)}{\cosh\left(\gamma \frac{d}{2}\right) - \gamma \frac{d}{2} \sinh\left(\gamma \frac{d}{2}\right)}, \\ \beta_2^{\text{ITC-2-1}} &= -d\left(1 - \tfrac{2}{\gamma d} \tanh(\gamma \tfrac{d}{2})\right), \end{aligned} \tag{9}$$

at least for straight sheets. For curved sheets the conditions ITC-2-1 include further terms including the curvature, and do not possess the relatively simple structure (7) anymore.

III. Boundary integral formulations

As the differential equation (3) is extended up to the interface Γ, we can represent the solution in $\mathbb{R}^2 \backslash \Gamma$ as [8]

$$\begin{aligned} e(\mathbf{x}) = &- \int_\Gamma G(\mathbf{x}-\mathbf{y})\, [\nabla e \cdot \mathbf{n}](\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) \\ &+ \int_\Gamma \mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{x}-\mathbf{y})\, [e](\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) + N(\mathbf{x}). \end{aligned} \tag{10}$$

where we use the Green's kernel $G(\mathbf{x}-\mathbf{y}) = -1/(2\pi)\log(|\mathbf{x}-\mathbf{y}|)$ and the Newton potential

$$N(\mathbf{x}) = \mathrm{i}\omega\mu_0 \int_{\mathbb{R}^2} G(\mathbf{x}-\mathbf{y})\, j_0(\mathbf{y})\, \mathrm{d}\mathbf{y}. \tag{11}$$

A. Boundary integral formulation for the ITCs of type I

The jump of the electric field vanishes and evaluating $e(\mathbf{x})$ in (10) for $\mathbf{x} \to \Gamma$ we obtain for the new unknown $\phi = [\nabla e \cdot \mathbf{n}]$

$$\{e\}(\mathbf{x}) = -\int_\Gamma G(\mathbf{x}-\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) + N(\mathbf{x}). \tag{12}$$

Multiplying this equation by β_1 and inserting (5a) gives the integral equation defining $\phi \in L^2(\Gamma)$

$$\phi(\mathbf{x}) + \beta_1 \int_\Gamma G(\mathbf{x}-\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) = \beta_1 N(\mathbf{x}). \tag{13}$$

The integral equation is of second kind, *i. e.*, the associated operator is the sum of identity and a compact pertubation which leads to system matrices for Galerkin boundary element methods with conditioning number bounded independently of the meshwidth. See [4] for another boundary integral equation for ITCs of type I.

B. Boundary integral formulation for the ITCs of type II

Taking the mean of the limits $\mathbf{x} \to \Gamma$ from the two sides and inserting (7) we get the mixed integral formulation of second kind for the two unknowns $\phi = [\nabla e \cdot \mathbf{n}]$ and $\psi = [e]$

$$\begin{aligned} &\phi(\mathbf{x}) + \beta_1 \int_\Gamma G(\mathbf{x}-\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) \\ &\qquad - \beta_1 \left\{ \int_\Gamma \mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{x}-\mathbf{y})\, \psi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) \right\} = \beta_1 N(\mathbf{x}), \end{aligned} \tag{14a}$$

$$\begin{aligned} &\psi(\mathbf{x}) + \beta_2 \left\{ \mathbf{n} \cdot \nabla_{\mathbf{x}} \int_\Gamma G(\mathbf{x}-\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) \right\} \\ &\qquad - \beta_2\, \mathbf{n} \cdot \nabla_{\mathbf{x}} \int_\Gamma \mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{x}-\mathbf{y})\, \psi(\mathbf{y})\, \mathrm{d}\Gamma(\mathbf{y}) = \beta_2 \nabla_{\mathbf{x}} N(\mathbf{x}) \cdot \mathbf{n}. \end{aligned} \tag{14b}$$

For the thin sheet conditions [3] and the impedance conditions ITC-2-1 there is a unique solution $(\phi, \psi) \in L^2(\Gamma) \times H^{1/2}(\Gamma)$.

IV. Asymptotic boundary element methods

We propose Galerkin boundary element methods for the weak form of (13) and (14), which are the equations multiplied with test functions and integrated over Γ. As test and trial functions for the ITCs of type I (13) we may use piecewise constants functions

$$S_0^{-1}(\Gamma_h) := \left\{ v_h \in L^2(\Gamma) : v_h \in \mathbb{P}_0(K_j), j = 1, \ldots, N_h \right\},$$

on a panelization Γ_h of Γ with N_h straight panels K_j of maximal length h or piecewise linear, continuous functions

$$S_1^0(\Gamma_h) := \left\{ v_h \in L^2(\Gamma) \cap C(\Gamma) : v_h \in \mathbb{P}_1(K_j), j = 1, \ldots, N_h \right\},$$

where the discretisation error in the $L^2(\Gamma)$ decays like $O(h)$ using $S_0^{-1}(\Gamma_h)$ and, if the solution has higher smoothness due to a smoothness of Γ, like $O(h^2)$ using $S_1^0(\Gamma_h)$. For the ITCs of type II (13) we may use either $S_0^{-1}(\Gamma_h)$ or $S_1^0(\Gamma_h)$ for ϕ and we are obliged to use $S_1^0(\Gamma_h)$ for ψ, where the convergence rates follow accordingly.

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