Efficient numerical algorythms on large scale magnetic field problems using an iterative domain decomposition method

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***Abstract*—An iterative domain decomposition method is applied to magnetostatic problems. In our previous methods the gauge condition is neglected, then the magnetic vector potential is only one unknown function. On the other hand, it has been well-known that some theoretical results has been introduced, where a mixed formulation with the Lagrange multiplier is introduced in order to impose the gauge condition. Therefore, in this paper, we formulate again an iterative domain decomposition method based on a mixed formulation of magnetostatic problem, and discuss relations with the previous one.**

***Index Terms*—numerical analysis, mathematical analysis, finite element methods, iterative algorithms, magnetostatics**

I. Introduction

We have introduced an iterative domain decomposition method to solve quite large scale electromagnetic field problems; see, for example, Kanayama *et al*. [7]. In our previous methods the gauge condition is neglected, then the magnetic vector potential is only one unknown function. These previous results focus themselves on the engineering points of view: the previous formulation enables us to reduce computational consts in practical large scale simulations. However this formulation yields an indeterminate linear system, it is difficult to mathematically justify numerical results, for example unique solvability of the problems and convergency of the approximate solution.

On the other hand, some theoretical results has been introduced by, for example, Kikuchi [5], [6], where a mixed formulation with the Lagrange multiplier is introduced in order to impose the gauge condition. These results focus themselves on the mathematical point of view: owing to the introduction of the Lagrange multiplier, their mixed formulation enable us to prove unique solvability of the problems and convergency of the approximate solution. However this formulation yields an indefinite linear system, it is difficult to find an appropriate iterative solver, which is efficient enough to reduce computational costs for practical large scale problems.

Now we formulate again an iterative domain decomposition method based on a mixed formulation of magnetostatic problem introduced in Kikuchi [5], [6], which enable us to prove unique solvability of the problems and convergency of the approximate solution. Moreover, to reduce computational costs, we simplify our iterative domain decomposition method into another one, and we discuss relations between the reduced formulation and the previous one.

II. Formulation of magnetostatic problems

Let Ω be a polyhedoral domain with its boundary Γ. Let $\boldsymbol{u}$ denote the magnetic vector potential, $\boldsymbol{f}$ an excitation current density, and ν the magnetic reluctivity. Following Kikuchi [5], a mixed weak formulation of magnetostatic problems with the Lagrange multiplier p is formulated as follows: given $\boldsymbol{f} \in (L^2(\Omega))^3$, find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), & \text{(1a)} \\ b(\boldsymbol{u}, q) = 0, \qquad \forall (\boldsymbol{v}, q) \in V \times Q, & \text{(1b)} \end{cases}$$

where V and Q denote functional spaces defined by $V := \{\boldsymbol{v} \in H(\mathrm{rot}; \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}$ and $Q := H^1_0(\Omega)$, respectively; $a(\,.\,,\,.\,)$ and $b(\,.\,,\,.\,)$ denote bilinear forms defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nu\, \mathrm{rot}\, \boldsymbol{u}\, \mathrm{rot}\, \boldsymbol{v}\, dx \quad \text{and} \quad b(\boldsymbol{v}, q) := \int_\Omega \boldsymbol{v}\, \mathrm{grad}\, q\, dx,$$

respectively; and $(\,.\,,\,.\,)$ denote an inner product of $L^2(\Omega)$.

III. Domain decomposition method

For simplicity, the domain Ω is assumed to be decomposed into two non-overlapping subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ with their boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$, respectively; and let γ_{12} be the interface between $\Omega^{(1)}$ and $\Omega^{(2)}$ defined by $\gamma_{12} := \overline{\Omega}^{(1)} \cap \overline{\Omega}^{(2)}$. Then, a two-subdomain problem is introduced by the followings: for $i = 1, 2$, find $(\boldsymbol{u}^{(i)}, p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ such that

$$\begin{cases} a^{(i)}(\boldsymbol{u}^{(i)}, \boldsymbol{v}^{(i)}) + b^{(i)}(\boldsymbol{v}^{(i)}, p^{(i)}) = (\boldsymbol{f}^{(i)}, \boldsymbol{v}^{(i)})_{\Omega^{(i)}}, & \text{(2a)} \\ b^{(i)}(\boldsymbol{u}^{(i)}, q^{(i)}) = 0, & \text{(2b)} \\ \boldsymbol{u}^{(1)} \times \boldsymbol{n} = \boldsymbol{u}^{(2)} \times \boldsymbol{n}, \quad p^{(1)} = p^{(2)} \quad \text{on } \gamma_{12}, & \text{(2c)} \\ a^{(2)}(\boldsymbol{u}^{(2)}, \bar{\boldsymbol{v}}^{(2)}(\boldsymbol{\eta})) + b^{(2)}(\bar{\boldsymbol{v}}^{(2)}(\boldsymbol{\eta}), p^{(2)}) \\ \quad = (\boldsymbol{f}^{(1)}, \bar{\boldsymbol{v}}^{(1)}(\boldsymbol{\eta}))_{\Omega^{(1)}} + (\boldsymbol{f}^{(2)}, \bar{\boldsymbol{v}}^{(2)}(\boldsymbol{\eta}))_{\Omega^{(2)}} \\ \qquad - a^{(1)}(\boldsymbol{u}^{(1)}, \bar{\boldsymbol{v}}^{(1)}(\boldsymbol{\eta})) - b^{(1)}(\bar{\boldsymbol{v}}^{(1)}(\boldsymbol{\eta}), p^{(1)}), & \text{(2d)} \\ b^{(2)}(\boldsymbol{u}^{(2)}, \bar{q}^{(2)}(\zeta)) = b^{(1)}(\boldsymbol{u}^{(1)}, \bar{q}^{(1)}(\zeta)), & \text{(2e)} \\ \qquad \forall (\boldsymbol{v}^{(i)}, q^{(i)}, \boldsymbol{\eta}, \zeta) \in V^{(i)} \times Q^{(i)} \times \Lambda \times \Xi. \end{cases}$$

Here, the superscripts denote functions and bilinear forms restricted into $\Omega^{(i)}$; function spaces $V^{(i)}_{\gamma_{12}}$, $Q^{(i)}_{\gamma_{12}}$, $V^{(i)}$, $Q^{(i)}$, Λ, and Ξ are defined by $V^{(i)}_{\gamma_{12}} := \{\boldsymbol{v} \in H(\mathrm{rot}; \Omega^{(i)});\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$

on $\partial\Omega^{(i)}\backslash\gamma_{12}\}$, $Q^{(i)}_{\gamma_{12}} := \{q \in H^1(\Omega^{(i)});\ \ q = 0 \text{ on } \partial\Omega^{(i)}\backslash\gamma_{12}\}$, $V^{(i)} := \{\boldsymbol{v} \in H(\mathrm{rot};\Omega^{(i)});\ \ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega^{(i)}\}$, $Q^{(i)} := H^1_0(\Omega^{(i)})$, $\Lambda := \{\boldsymbol{\lambda} : \gamma_{12} \to \mathbb{R}^3;\ \boldsymbol{\lambda} = (\boldsymbol{u}\times\boldsymbol{n})|_{\gamma_{12}},\ \boldsymbol{u} \in V\}$, and $\Xi := \{\xi : \gamma_{12} \to \mathbb{R};\ \xi = p|_{\gamma_{12}},\ p \in Q\}$, respectively; and set $\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})$ by any extension operator from Λ to $V^{(i)}_{\gamma_{12}}$ such that $\boldsymbol{\eta} = (\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})\times\boldsymbol{n})|_{\gamma_{12}}$, and $\overline{p}^{(i)}(\zeta)$ by any extension operator from Ξ to $Q^{(i)}_{\gamma_{12}}$ such that $\zeta = p(\zeta)|_{\gamma_{12}}$. A characterization of *tangential trace* spaces Λ and an *tangential extension* operator on $\overline{\boldsymbol{u}}^{(i)}(\boldsymbol{\eta})$ has been given in Alonso–Valli [1], Buffa–Ciarlet [2], Buffa, *et al.* [3], and Quarteroni–Valli [8].

The equivalency between both formulations and unique solvability could be obtained as follows:

Theorem 1: There exists a unique solution $(\boldsymbol{u}^{(i)}, p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ of the two-subdomain problem (2). Moreover, the one-domain problem (1) and the two-subdomain problem (2) are equivalent.

For $i = 1, 2$, let $\mathscr{E}^{(i)}(\boldsymbol{f}, \boldsymbol{\lambda}, \xi)$ an extention operator from $(L^2(\Omega))^3 \times \Lambda \times \Xi$ to $V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ defined by $\mathscr{E}^{(i)}(\boldsymbol{f}, \boldsymbol{\lambda}, \xi) := (\boldsymbol{u}^{(i)}, p^{(i)})$, where $(\boldsymbol{u}^{(i)}, p^{(i)})$ is the solution of the following magnetostatic problem:

$$\begin{cases} a^{(i)}(\boldsymbol{u}, \boldsymbol{v}) + b^{(i)}(q, p) = (\boldsymbol{f}, \boldsymbol{v})_{\Omega^{(i)}}, & \text{(3a)} \\ b^{(i)}(\boldsymbol{u}, q) = 0, \qquad \forall (v, q) \in V^{(i)} \times Q^{(i)}, & \text{(3b)} \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{\lambda}, \quad p = \xi \qquad \text{on } \gamma_{12} & \text{(3c)} \end{cases}$$

Then, a *Steklov–Poincaré* operator $\mathscr{A}$ from $\Lambda \times \Xi$ to $(\Lambda \times \Xi)'$ is set by

$$\langle \mathscr{A}(\boldsymbol{\lambda}, \xi), (\boldsymbol{\eta}, \zeta)\rangle := \sum_{i=1}^{2} \{ a^{(i)}(\overline{\boldsymbol{u}}^{(i)}, \overline{\boldsymbol{v}}^{(i)}) + b^{(i)}(\overline{\boldsymbol{v}}^{(i)}, \overline{p}^{(i)}) + b^{(i)}(\overline{\boldsymbol{u}}^{(i)}, \overline{q}^{(i)})\}, \qquad \forall \boldsymbol{\lambda}, \boldsymbol{\eta} \in \Lambda,\ \forall \xi, \zeta \in \Xi, \quad (4)$$

where $(\overline{\boldsymbol{u}}^{(i)}, \overline{p}^{(i)}) := \mathscr{E}^{(i)}(0, \boldsymbol{\lambda}, \xi)$ and $(\overline{\boldsymbol{v}}^{(i)}, \overline{q}^{(i)}) := \mathscr{E}^{(i)}(0, \boldsymbol{\eta}, \zeta)$; and an *interface source* $\boldsymbol{\chi} \in (\Lambda \times \Xi)'$ is set by

$$\langle \boldsymbol{\chi}, (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} := \sum_{i=1}^{2} \{ (\boldsymbol{f}^{(i)}, \overline{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} - a^{(i)}(\widehat{\boldsymbol{u}}^{(i)}, \overline{\boldsymbol{v}}^{(i)}) - b^{(i)}(\overline{\boldsymbol{v}}^{(i)}, \widehat{p}^{(i)}) - b^{(i)}(\widehat{\boldsymbol{u}}^{(i)}, \overline{q}^{(i)})\}, \quad \forall \boldsymbol{\eta} \in \Lambda,\ \forall \zeta \in \Xi, \quad (5)$$

where $(\widehat{\boldsymbol{u}}^{(i)}, \widehat{p}^{(i)}) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \boldsymbol{0}, 0)$. Now we introduce the following *interface problem* on γ_{12}:

$$\langle \mathscr{A}(\boldsymbol{\lambda}, \xi), (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} = \langle \boldsymbol{\chi}, (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}}, \quad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi. \quad (6)$$

By using the solution $(\boldsymbol{u}^{(i)}, p^{(i)})$ of two-subdomain problem (2), let us set $(\boldsymbol{\lambda}, \xi)$ by $\boldsymbol{\lambda} := \boldsymbol{u}^{(1)}\times\boldsymbol{n}\ (= \boldsymbol{u}^{(2)}\times\boldsymbol{n})$ and $\xi := p^{(1)}(= p^{(2)})$. Then, because of (2d)–(2e), $(\boldsymbol{\lambda}, \xi)$ satisfies the *interface problem* (6). On the other hand, once the solution $(\boldsymbol{\lambda}, \xi)$ is obtained by solving the *interface problem* (6), for $i = 1, 2$, each pair $(\boldsymbol{u}^{(i)}, p^{(i)}) \in V^{(i)}_{\gamma_{12}} \times Q^{(i)}_{\gamma_{12}}$ could be found from the problem (2a) and (2b) in the corresponding subdomain $\Omega^{(i)}$, where the solution $(\boldsymbol{\lambda}, \xi)$ is regarded as the Dirichlet boundary on the interface: $\boldsymbol{u}^{(i)}\times\boldsymbol{n} = \boldsymbol{\lambda}$ and $p^{(i)} = \xi$ on γ_{12}. Finally, we can obtain the solution $(\boldsymbol{u}, p)$ of the one-domain problem (1).

Therefore, error analysis of the approximate solution of the problem (2) could be reduced into error analysis of the one of each subdomain problem; For example, when magnetic vector potential $\boldsymbol{u}$ is approximated by the Nedelec element of the first order and the Lagrange multiplier p is approximated by the conventional $P1$-element.

The *interface problem* (6) is symmetric, and not positive definite. Moreover, if $\boldsymbol{f}^{(i)}$ satisfies that $\mathrm{div}\,\boldsymbol{f}^{(i)} = 0$ in $\Omega^{(i)}$, then $p^{(i)}$ vanishes. These facts implies that, by following Glowinski *et al.* [4], the following reduced conjugate gradient algorithm could be obatined, where the Lagrange multiplier is not required in the iterative procedure of actual computation:

Choose $\boldsymbol{\lambda}_0$;
Compute $\boldsymbol{g}_0$ **by** (7);
$\boldsymbol{w}_0 := \boldsymbol{g}_0$;
for $k = 0, 1, \ldots$;
  Compute $\mathscr{A}_1(\boldsymbol{w}_k, 0)$ **by** (8);
  $\alpha_k := (\boldsymbol{g}_k, \boldsymbol{g}_k)/(\mathscr{A}_1(\boldsymbol{w}_k, 0), \boldsymbol{w}_k,)$;
  $\boldsymbol{\lambda}_{k+1} := \boldsymbol{\lambda}_k - \alpha_k \boldsymbol{w}_k$;
  $\boldsymbol{g}_{k+1} := \boldsymbol{g}_k - \alpha_k \mathscr{A}_1(\boldsymbol{w}_k, 0)$;
  $\beta_k := (\boldsymbol{g}_{k+1}, \boldsymbol{g}_{k+1})/(\boldsymbol{g}_k, \boldsymbol{g}_k)$;
  If $(\boldsymbol{g}_{k+1}, \boldsymbol{g}_{k+1})/(\boldsymbol{g}_0, \boldsymbol{g}_0) < \varepsilon$, **break**;
  $\boldsymbol{w}_{k+1} := \boldsymbol{g}_{k+1} + \beta_k \boldsymbol{w}_k$;
end;

In the reduced conjugate gradient algorythm, $\boldsymbol{g}_0$ could be computed by the first component of the following equation:

$$\langle (\boldsymbol{g}_0, \delta_0), (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} = \sum_{i=1}^{2} \{ a^{(i)}(\widetilde{\boldsymbol{u}}^{(i)}_0, \overline{\boldsymbol{v}}^{(i)}) + b^{(i)}(\overline{\boldsymbol{v}}^{(i)}, \widetilde{p}^{(i)}_0) - (\boldsymbol{f}^{(i)}, \overline{\boldsymbol{v}}^{(i)})_{\Omega^{(i)}} + b^{(i)}(\widetilde{\boldsymbol{u}}^{(i)}_0, \overline{q}^{(i)})\}, \quad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (7)$$

where $(\widetilde{\boldsymbol{u}}^{(i)}_0, \widetilde{p}^{(i)}_0) := \mathscr{E}^{(i)}(\boldsymbol{f}^{(i)}, \boldsymbol{\lambda}_0, 0)$; and $\mathscr{A}_1(\boldsymbol{w}_k, 0)$ could be computed by the first component of the following equation:

$$\langle \mathscr{A}(\boldsymbol{w}_k, 0), (\boldsymbol{\eta}, \zeta)\rangle_{\gamma_{12}} = \sum_{i=1}^{2} \{ a^{(i)}(\widehat{\boldsymbol{u}}^{(i)}_k, \overline{\boldsymbol{v}}^{(i)}) + b^{(i)}(\overline{\boldsymbol{v}}^{(i)}, \widehat{p}^{(i)}_k) + b^{(i)}(\widehat{\boldsymbol{u}}^{(i)}_k, \overline{q}^{(i)})\}, \quad \forall (\boldsymbol{\eta}, \zeta) \in \Lambda \times \Xi, \quad (8)$$

where $(\widehat{\boldsymbol{u}}^{(i)}_k, \widehat{p}^{(i)}_k) := \mathscr{E}^{(i)}(\boldsymbol{0}, \boldsymbol{w}_k, 0)$.

This algorythm is based on abstract mathematical analysis. Therefore, we can easily extend the algorythm into other larger scale computations of magnetic field problems, for example, eddy current problems, etc.

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