

A Note on Faraday Paradoxes

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Abstract—Can the voltage at the terminal of a moving conductive loop always be calculated with the Faraday law in global form, $U = -d\Phi/dt$? A WWW search on "Faraday Paradoxes" reveals that this seemingly simple question has led to confusions in a class of induction problems where slip-rings are present. The Faraday law has been said to be applicable only when the conductive loop's "material identity is preserved by its movement". Even though the answers found by such arguments are by and large correct, many statements found in the literature are ambiguous and misleading. Basing ourselves on the universality of the Faraday law in connection with Ohm's law for moving media, we hope to contribute to the explanation of Faraday Paradoxes in a straightforward and insightful manner.

Index Terms—Electrical engineering education, Physics education

I. INTRODUCTION

The Faraday law in global form, $U = -\frac{d\Phi}{dt}$, is generally used to calculate induction phenomena, where Φ is the magnetic flux through a surface \mathcal{A} , and U is the voltage induced along its boundary $\partial\mathcal{A}$. Consider the conductor loop shown in Fig. 1 (right) which coincides with $\partial\mathcal{A}$ and has its connection terminal in $\mathcal{P} \in \partial\mathcal{A}$. The terminal voltage is often stated to be $U_{12} = -\frac{d\Phi}{dt}$, provided the loop resistance is sufficiently small as compared to the voltmeter input resistance, and the resulting circuit diagram is drawn in the load convention. In case of a moving loop in an external magnetic field, the *convective derivative* [1] must be employed to compute $\frac{d\Phi}{dt}$, that is,

$$\begin{aligned} -\frac{d\Phi}{dt} &= -\frac{d}{dt} \int_{\mathcal{A}} \vec{B} \cdot d\vec{a} \\ &= -\int_{\mathcal{A}} \frac{\partial}{\partial t} \vec{B} \cdot d\vec{a} + \int_{\partial\mathcal{A}} (\vec{v}_p \times \vec{B}) \cdot d\vec{r}, \end{aligned}$$

where \vec{v}_p denotes the velocity of the integration path, i.e., of the rim of the surface $\partial\mathcal{A}$. The two factors on the right-hand side are called *transformer EMF* (Electro-Motive Force) and *generator EMF*, respectively.

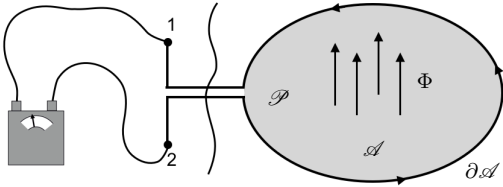


Fig. 1. Abstract setting of the problem. The surface \mathcal{A} is the gray shaded surface with boundary $\partial\mathcal{A}$. A voltmeter is connected at the terminal of a conductive loop, coinciding with the surface boundary.

Some textbooks issue a warning on the above statements: they may hold only when the *material identity* of the loop is

guaranteed [2]. In Feynman [3] we read: *When the material in a circuit is changing we must return to the basic laws. The correct physics is always given by the two basic laws*

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}), \quad \text{curl} \vec{E} = -\frac{\partial}{\partial t} \vec{B}.$$

But does this imply that the global Faraday law is less general and yields wrong results in some circumstances? To answer this question, let us first consider the following experiments:

- 1) The stretched-wire magnetic measurement [4] is based on a conductive wire (tensioned with negligible sag) displaced in the aperture of a magnet, which is free of any conducting or permeable material; see Fig. 2 (left). A conceptually easy situation is given when any sliding contact, connector, voltmeter, or auxiliary wire is in the field-free region. Then the induced voltage is proportional to the field integral times the velocity and the integrated voltage is proportional to the flux intercepted by the surface spanned during the displacement.

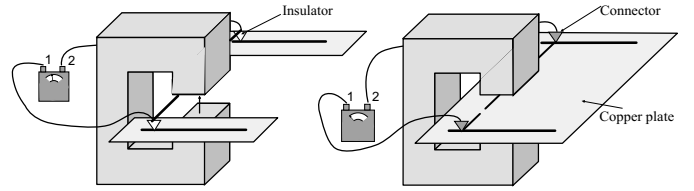


Fig. 2. Left: Stretched-wire measurement. The wire itself is insulated with respect to the sliders. Right: An example from Feynman adapted to our situation (conducting plate in static field). The sliders are moved on the conducting plate from inside out. No voltage is induced.

- 2) An example due to Feynman [3] is slightly adjusted to compare it with the setup for the stretched-wire measurement. According to Fig. 2 (right), a conducting plate is placed in the aperture of a C-shaped magnet. As the sliding contacts are moved across the plate, there is no EMF although we have a large change of flux through the surface spanned by $\partial\mathcal{A}$.
- 3) The homopolar generator [5] shown schematically in Fig. 3 produces an EMF even though neither the integration path is moving, nor the field is changing.
- 4) When a coil is moved through the aperture of a C-shaped magnet, see Fig. 4 (top), an EMF is induced similarly to the case of the stretched-wire measurement above. In the Hering experiment [6], as depicted in Fig. 4 (bottom), the coil is replaced by a loop that consists of two flexible leads with sliding contacts, no EMF is induced when the contacts are pulled across the iron yoke, even though there is a varying flux linkage in the loop.

II. SYSTEMATIC RESOLUTION OF THE PARADOXES

Although the above experiments are all related to induction problems, only the stretched-wire measurement and the coil in the first part of the Hering experiment are covered by the simple application of $U_{12} = -\frac{d\Phi}{dt}$. Should we therefore deduce that the global Faraday law is not universal? The global Faraday law $U(\partial\mathcal{A}) = -\frac{d\Phi}{dt}$ is related to the Maxwell equations $\vec{E} = -\text{grad}\varphi - \frac{\partial\vec{A}}{\partial t}$ via $\Phi = \int_{\mathcal{A}} \vec{B} \cdot d\vec{a} = \int_{\partial\mathcal{A}} \vec{A} \cdot d\vec{r}$ and the convective derivative. Global and local descriptions are equally valid. However, the above formula $U_{12} = -\frac{d\Phi}{dt}$ is not a direct consequence of the global Faraday law. As it relates U_{12} to $U(\partial\mathcal{A})$, it makes use of the constitutive relation, i.e., Ohm's law and the resistive voltage-divider rule. This may appear to be insignificant at first glance, but it holds the key to all of the above apparent paradoxes. Recall that Ohm's law in moving media reads

$$\rho \vec{J} = \vec{E} + \vec{v}_m \times \vec{B},$$

where ρ is the resistivity and \vec{v}_m the velocity of the conductive medium. This version of Ohm's law is a direct consequence of the Lorentz force $\vec{F} = Q(\vec{E} + \vec{v}_m \times \vec{B})$, where Q is the charge-carrier in the conductor moving through a field (\vec{E}, \vec{B}) . Writing \vec{v}_p for the velocity field of the integration path $\partial\mathcal{A}$, we follow Feynman's suggestion by applying Ohm's law and $\text{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ to the convective derivative; the terminal voltage is obtained from the voltage divider rule,

$$\begin{aligned} -\frac{d\Phi}{dt} &= U(\partial\mathcal{A}) \\ &= -\int_{\mathcal{A}} \frac{\partial\vec{B}}{\partial t} \cdot d\vec{a} + \int_{\partial\mathcal{A}} (\vec{v}_p \times \vec{B}) \cdot d\vec{r} \\ &= \int_{\mathcal{A}} \text{curl}\vec{E} \cdot d\vec{a} + \int_{\partial\mathcal{A}} (\vec{v}_p \times \vec{B}) \cdot d\vec{r} \\ &= \int_{\partial\mathcal{A}} \vec{E} \cdot d\vec{r} + \int_{\partial\mathcal{A}} (\vec{v}_p \times \vec{B}) \cdot d\vec{r} \\ &= U_{12} + \int_{\partial\mathcal{A} \setminus \mathcal{P}} \rho \vec{J} \cdot d\vec{r} + \int_{\partial\mathcal{A}} ((\vec{v}_p - \vec{v}_m) \times \vec{B}) \cdot d\vec{r}. \end{aligned}$$

For simplification, and without loss of generality, we set $\rho = 0$ on $\partial\mathcal{A} \setminus \mathcal{P}$, that is, the path resistance is small compared to the input resistance of the voltmeter. We find

$$U_{12} = -\frac{d\Phi}{dt} - \int_{\partial\mathcal{A}} ((\vec{v}_p - \vec{v}_m) \times \vec{B}) \cdot d\vec{r}$$

and, alternatively,

$$U_{12} = -\int_{\mathcal{A}} \frac{\partial\vec{B}}{\partial t} \cdot d\vec{a} + \int_{\partial\mathcal{A}} (\vec{v}_m \times \vec{B}) \cdot d\vec{r}.$$

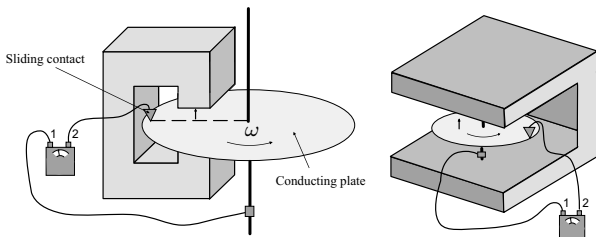


Fig. 3. Homopolar generator. Left: Nonuniform field on the disc as in Faraday's original experiment. Right: Rotating disc in uniform magnetic field.

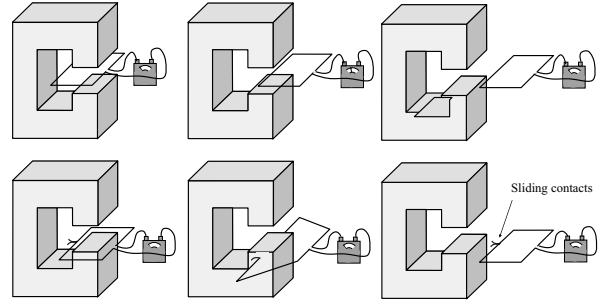


Fig. 4. Illustration of the Hering experiment.

All of the above seemingly paradoxical experiments can now be resolved by this equation, which differs from the statement in the introduction only by the distinction between path velocity and material velocity:

- 1) A stretched-wire measurement of a magnet with a pole length L and a uniform pole field B_0 is described by $\vec{v}_p = \vec{v}_m = \vec{v}$ and $\frac{\partial\vec{B}}{\partial t} = 0$. The terminal voltage is $U_{12} = \int_{\partial\mathcal{A}} (\vec{v} \times \vec{B}) \cdot d\vec{r} = vLB_0$.
- 2) For the Feynman-plate experiment, $\vec{v}_m = 0$ and $\frac{\partial\vec{B}}{\partial t} = 0$. Therefore $U_{12} = 0$.
- 3) The case of the homopolar generator, Fig. 3 (right), with a disc of radius r_0 rotating at an angular velocity ω , $v_m = r\omega$, in a homogeneous field B_0 , yields $\vec{v}_p = 0$, $\frac{\partial\vec{B}}{\partial t} = 0$, and $U_{12} = \int_{\partial\mathcal{A}} (\vec{v}_m \times \vec{B}) \cdot d\vec{r} = \frac{1}{2}\omega r_0^2 B_0$.
- 4) For Hering's experiment, Fig. 4 (bottom), we find $\vec{v}_m = 0$, $\frac{\partial\vec{B}}{\partial t} = 0$, and therefore $U_{12} = 0$.

III. CONCLUSION

The global Faraday law for moving paths with velocity \vec{v}_p are valid under all technical circumstances. Misinterpretations of the law may result from the assumption that the terminal voltage U_{12} can be determined by the global Faraday law alone. The voltage divider rule naturally introduces Ohm's law for conductive media moving with material velocity \vec{v}_m . The naive application of $U_{12} = -\frac{d\Phi}{dt}$ yields correct results only if $\vec{v}_p = \vec{v}_m$, that is, the boundary $\partial\mathcal{A}$ in the Faraday law coincides with a thin conducting wire. It should be noted that the velocities \vec{v}_p and \vec{v}_m , as well as the fields \vec{E} , \vec{B} , and \vec{J} are expressed with respect to one and the same observer. In the full paper we plan to elaborate on the notion of Eulerian and Lagrangian observers, and the independence of the above findings of a specific viewpoint.

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