

# Computed Basis Functions and the Nonconforming Voxel Finite Element Method

M. Nazari, J. P. Webb

Department of Electrical and Computer Engineering, McGill University, H3A 0E9, Montreal, Canada  
 moein.nazarirastehkenari@mail.mcgill.ca, jon.webb@mcgill.ca

**Abstract**—The nonconforming voxel finite element method (NVFEM) is an alternative to conventional mesh generation that uses purely rectangular elements. Its main drawback is the need to approximate boundaries that would otherwise pass through the interior of the elements. By using computed basis functions, boundaries inside elements are permitted. Results for a magnetostatic problem demonstrate the improved accuracy of the method, compared to conventional NVFEM.

**Index Terms**—Finite element methods, magnetostatics, adaptive mesh refinement.

## I. INTRODUCTION

The nonconforming voxel finite element method (NVFEM) is an alternative to conventional finite element (FE) mesh generation, proposed by Odawara *et al.* [1] for analysis of magnetic field problems. The problem domain is recursively subdivided into cubes, and the cubes that are undivided at the end are taken to be the finite elements. Compared to traditional meshing, the strengths of the method are its simplicity, efficiency, robustness and suitability for use with multigrid schemes [2].

It has, however, one drawback: the FE mesh does not match the problem geometry, meaning that the boundaries and material interfaces pass through the interior of the elements. The true interfaces have to be approximated in a “staircase” fashion so that each element lies on one side of the interface or the other, but this leads to a loss of accuracy. There has been an attempt to remedy this by adjusting the material properties within elements straddling an interface [3]. We propose an alternative: the use of elements that permit material discontinuities within them, sometimes called “composite” [4][5]. The specific technique we use is called the method of computed basis functions (CBFs) [6][7].

## II. THE NONCONFORMING VOXEL FEM

We consider the 2D version of the method. We solve a differential equation of the form  $Du = f$  for an unknown potential  $u$ . The partial differential operator  $D$  depends on the the material properties at each point.

Place the problem domain,  $\Omega$ , in a rectangular box. There are a number of regions within  $\Omega$ , each containing a different material. Subdivide the box initially into  $M^2$  equal boxes. Consider each in turn. If a box is entirely inside or entirely outside each region, it contains only one material and is designated “uniform”. Otherwise, subdivide it into four equal smaller boxes, and so on. Continue this for  $L$  levels of subdivision, i.e., until all the non-uniform boxes are the result of  $L$  subdivisions.

During this process there will arise “hanging” nodes, i.e., nodes that lie on the edge of an undivided box, rather than at one of its corners. If, during the subdivision, a second hanging node is placed on the edge of a box, that box is itself subdivided, even it is uniform. In this way, the final mesh does not have any “second-level” hanging nodes. Each remaining hanging node is handled by constraining its potential to be the average of the potentials of the nodes at the ends of its edge.

At the end of this process the  $N$  resulting boxes, which are of various sizes, are taken as finite elements (Fig. 1(a), (b)). The majority of the elements will, by construction, be uniform and can be processed in the usual way, employing the standard basis functions of such an element [8].

However, some of them – clustered around the boundaries of the regions – are nonuniform. In conventional NVFEM, standard basis functions are used for these elements too, but it is more accurate to compute special basis functions that take into account the material discontinuities within the element.

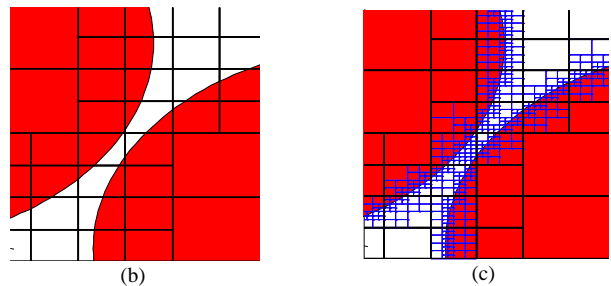
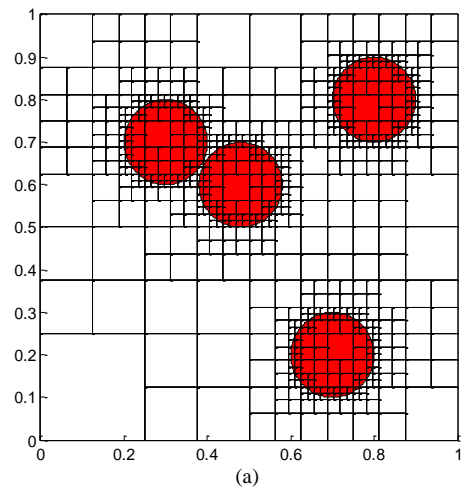


Fig. 1 (a) Mesh of the magnetostatic problem (NVFEM;  $M=4$ ,  $L=4$ ); (b) enlarged view (NVFEM); (c) enlarged view (CBF-cells,  $L_c=3$ )

### III. COMPUTED BASIS FUNCTIONS

In [6][7] a CBF method was described that employs a finite difference grid within each element to compute the basis functions. The grid is identical, and regular, in every element of the mesh. Here, we use a different approach that is more suited to the nature of the problem.

Continue to subdivide the nonuniform elements as before, for a further  $L_c$  levels of subdivision. This time, allow arbitrary-level hanging nodes when these arise on edges of elements. A box at the end of this process that was not present at the beginning (Fig. 1(c)) is called a ‘‘CBF-cell’’.

Consider the set of edges of nonuniform elements. We build two ‘‘edge functions’’ per edge, each of which has the value 1 at one end of the edge and 0 at the other. The edge functions are found by solving  $Du = 0$  as a 1D FE problem, taking into account the varying material property along the edge. The 1D mesh corresponds to the edges of the CBF-cells.

Now consider each nonuniform element. Four CBFs are found by solving  $Du = 0$  using FEM four times. Each CBF takes the value 1 at one of the nodes of the element and 0 at the other three nodes. The previously-computed edge functions provide the boundary conditions, so the FE problem is entirely local, i.e., confined just to one element. The FE mesh for an element is defined by the CBF-cells within it.

Since the basis functions computed with this procedure solve  $Du = 0$  approximately, they will tend to satisfy the physically correct changes in gradient across the material interfaces within the element and will give global solutions that respect the region boundaries. For fixed  $L_c$ , the computational cost of computing all the basis functions and assembling the  $N$  FE local matrices is  $O(N)$ , the same as it is for conventional matrix assembly.

### IV. NUMERICAL RESULTS

In this section we compare the performance of the proposed method with conventional NVFEM. For this purpose, we use a magnetostatic problem and compute the magnetic scalar potential  $\psi$ . We solve the Poisson equation  $\nabla \cdot \mu_r \nabla \psi = 0$  in a  $1\text{ m} \times 1\text{ m}$  domain  $\Omega$  filled with air, containing four cylinders of radius  $r = 0.1\text{ m}$  and relative permeability  $\mu_r = 10$  (Fig. 1). The Dirichlet boundary conditions  $\psi = 1\text{ Am}^{-1}$  and  $\psi = 0$  are imposed on the upper and lower edges of the domain, respectively, creating a magnetic field that, in the absence of the cylinders, would be uniform and vertical.

Fig. 1(a) and (b) illustrate the mesh for the NVFEM. As expected, most of the refinement has occurred at the boundaries of the cylinder. To apply our method, local FE problems for all nonuniform elements are constructed. Fig. 1(c) shows the CBF-cells of the local FE problems.

The problem is solved in three ways: using conventional NVFEM, using the proposed method in which the nonuniform elements are subdivided three times ( $L_c = 3$ ) and using the proposed method with  $L_c = 6$ . The errors in the magnetostatic stored energy in all three cases are shown in Fig. 2. The number of elements is increased by increasing  $M$ , keeping  $L$  and  $L_c$  fixed. The reference energy was obtained using

commercial FE software [9], with a mesh of 306,177 fourth-order triangular elements and is estimated to be accurate to 0.001%. It can be seen from these plots that a solution at a specified precision can be obtained with the proposed method with many fewer degrees of freedom than with NVFEM. For example, an error of 0.06% can be achieved with  $L_c = 6$  with about  $N = 3760$ , which is 9.6 times smaller than with conventional NVFEM.

### V. CONCLUSION

By taking into account the effect of a material interface within a finite element, CBFs can greatly increase the accuracy of NVFEM.

### REFERENCES

- [1] S. Odawara, Y. H. Gao, K. Muramatsu, T. Okitsu, and D. Matsushashi, ‘‘Magnetic Field Analysis Using Voxel Modeling With Nonconforming Technique,’’ *IEEE Transactions on Magnetics*, vol. 47, pp. 1442-1445, May 2011.
- [2] S. Odawara, Y. H. Gao, and K. Muramatsu, ‘‘Magnetic Field Analysis With Nonconforming Voxel Modeling Using the Nested Geometric Multigrid Method,’’ *IEEE Transactions on Magnetics*, vol. 48, pp. 595-598, Feb 2012.
- [3] K. Watanabe, Y. Iijima, K. Kawano, and H. Igarashi, ‘‘Voxel Based Finite Element Method Using Homogenization,’’ *IEEE Transactions on Magnetics*, vol. 48, pp. 543-546, Feb 2012.
- [4] S. Sauter, R. Warnke, ‘‘Composite finite elements for elliptic boundary value problems with discontinuous coefficients,’’ *Computing*, vol. 77, pp. 29-55, Feb 2006.
- [5] F. Liehr, T. Preusser, M. Rumpf, S. Sauter, and L. Schwen, ‘‘Composite finite elements for 3D image based computing,’’ *Computing and Visualization in Science*, vol. 12, pp. 171-188, 2009.
- [6] H. Gu, J. Gotman, and J. P. Webb, ‘‘Finite-Element Simulation in Bioelectromagnetics Without the Need for Modeling and Meshing,’’ *IEEE Transactions on Magnetics*, vol. 45, pp. 1678-1681, Mar 2009.
- [7] H. Gu, J. Gotman and J. P. Webb, ‘‘Computed Basis Functions for Finite Element Analysis Based on Tomographic Data,’’ *IEEE Transactions on Biomedical Engineering*, vol. 58, pp. 2498-2505, Sep 2011.
- [8] O. C. Zienkiewicz and R. L. Taylor, *The finite element method* vol. 1: McGraw-Hill, 1989.
- [9] ElecNet 7.3, 2012, Infolytica Corporation, Montreal, Canada, <http://www.infolytica.com>.

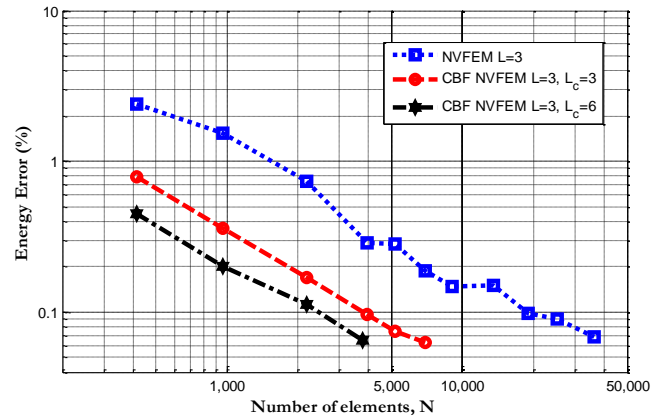


Fig. 2 Error in energy versus number of elements, conventional NVFEM in comparison with the proposed method