# Efficient Numerical Integration for Post-processing and Matrix Assembly of Finite Element Subdomains

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***Abstract*—The efficiency of numerical integration methods for finite element analysis (FEA) is investigated. The focus of this paper, within the context of traditional FEA, is the post-processing operation of numerical integration over a subdomain containing potentially large numbers of elements. Improvements to the efficiency of such operations directly increase the efficacy of the assembly process for the mass and stiffness matrices of a generalized family of macro elements. The analogous post-processing and matrix assembly methods are refined such that repeated computations are eliminated, effectively treating a large number of traditionally separate numerical integrations as one full-domain integral. Electromagnetic applications are tested to demonstrate the efficiency of the method. Results confirm a ten-fold reduction in computational cost for a range of applications.**

***Index Terms*—Computer aided analysis, scientific computing, computational electromagnetics, finite element methods.**

## I. Introduction

Finite element methods (FEM) for electromagnetic problems rely heavily on accurate and efficient numerical integration schemes in both the assembly, and post-processing phases of analysis [1]. The objective of this paper is to describe and illustrate the performance of a new technique developed to improve the efficiency of the integration of field solutions, as well as the assembly of the local mass and stiffness matrices for a generalized family of macro elements, which require numerical integration over potentially large collections of elements [2]. The method is shown for two-dimensional field solutions on curvilinear triangular meshes and compared to a "classical" numerical integration approach. The accuracy and efficiency of both methods are assessed and compared. It is shown that for the same level of accuracy, the new technique is computationally less expensive, with gains increasing as the number of elements increases.

## II. Theory

Consider the integral $I = \iint_\Omega \phi(x,y)\,dxdy$, where $\Omega$ is the domain of a two-dimensional Helmholtz problem, and $\phi$ is the FEM solution to the problem. This equation can be evaluated numerically by integrating element by element and summing the results. This is accomplished either analytically (using the same concepts employed to assemble universal matrices [3]), or by using numerical quadrature schemes (for example, by using [4] for the abscissae and weights for the triangle). The second approach has been favored in element assembly for its ability to deal with non-linear materials and curvilinear element geometries, and will be the focus of this work.

Expanding the solution $\phi$ in terms of its corresponding weighted sum of basis functions over each element and using an appropriate quadrature rule, the integral can be rewritten as

$$I \cong \sum_{e=1}^{n} \left\{ \sum_{j=1}^{m} \left[ \left( \sum_{i=1}^{l} N_i(\xi_j,\eta_j)\phi_i^e \right) \left| \boldsymbol{J}^e(\xi_j,\eta_j) \right| \right] w_j \right\}, \quad (1)$$

where the domain $\Omega$ is comprised of $n$ elements, each of which has $l$ basis functions $N_i(\xi,\eta)$ with associated field values $\phi_i^e$ and Jacobian determinant $|\boldsymbol{J}^e(\xi,\eta)|$ (corresponding to the transformation from global $xy$-space to simplex $\xi\eta$-space). Each element is evaluated at the $m$ quadrature abscissae $(\xi_j,\eta_j)$ and multiplied by their respective weights $w_j$. Superscript $e$ denotes element-dependent quantities.

For interpolatory [3] or hierarchal [5] bases, all elements share the same basis functions regardless of element geometry (in the hierarchal case, lower order elements can be interpreted as higher order elements with zero-valued higher order basis functions). Therefore, permuting the "classical" ordering of the operations in (1) can save function evaluations when performing numerical integration over collections of elements. Rearranging (1) yields

$$I \cong \sum_{j=1}^{m} \left\{ \sum_{i=1}^{l} \left[ N_i(\xi_j,\eta_j) \left( \sum_{e=1}^{n} \phi_i^e \left| \boldsymbol{J}^e(\xi_j,\eta_j) \right| \right) \right] \right\} w_j, \quad (2)$$

which we call the "stacking" method, since the manipulations can be interpreted as transforming a set of $n$ integrands over $n$ element domains into one integrand over a common domain.

If we restrict our discussion to planar curvilinear triangles with quadratic edges, as described in [1], the Jacobian determinant $|\boldsymbol{J}^e(\xi,\eta)|$ in (2) can be written as

$$|\boldsymbol{J}^e(\xi,\eta)| = a_1^e + a_2^e\xi + a_3^e\eta + a_4^e\xi^2 + a_5^e\xi\eta + a_6^e\eta^2, \quad (3)$$

where the coefficients $a_t^e$ for $t \in \mathbb{N}$ ranging from 1 to 6 are determined by the positions of the geometric vertex and mid-side nodes of each element [6]. Then (2) can be vectorized as

$$I \cong \operatorname{diag}(\boldsymbol{N\Phi J}_a \boldsymbol{J}_b) \cdot \boldsymbol{w}, \quad (4)$$

where $\boldsymbol{N} \in \mathbb{R}^{m\times l}$ with entries $N_{ij} = N_j(\xi_i,\eta_i)$, $\boldsymbol{\Phi} \in \mathbb{R}^{l\times n}$ with $\Phi_{ij} = \phi_i^{(j)}$, $\boldsymbol{J}_a \in \mathbb{R}^{n\times 6}$ with $(J_a)_{ij} = a_j^{(i)}$, $\boldsymbol{J}_b \in \mathbb{R}^{6\times m}$ with $(J_b)_j = \begin{bmatrix} 1 & \xi_j & \eta_j & \xi_j^2 & \xi_j\eta_j & \eta_j^2 \end{bmatrix}^T$ its $j^{th}$ column, and $\boldsymbol{w} \in \mathbb{R}^{m\times 1}$.

Two particularly interesting post-processing operations which are also associated with constructing the stiffness and mass matrix entries for the generalized macro elements as described in [2] – where $\phi_p$ and $\phi_q$ represent two different piecewise-defined subdomain basis functions – are given by

$$S_{pq} = \int_\Omega \nabla\phi_p \cdot \nabla\phi_q \, dxdy, \qquad M_{pq} = \int_\Omega \phi_p\phi_q \, dxdy, \quad (5)$$

over the mesh of curvilinear triangles used to discretize the subdomain Ω. To use one such generalized macro element in a larger mesh, computing the entries (5) can be done using generalizations of the "stacking" method as follows.

Consider the diagonal entries of the mass matrix (the off-diagonal terms follow the same development, but they have been omitted for brevity and clarity). Since $\phi_p$ and $\phi_q$ are both piecewise-defined over each constituent element, we have

$$M \cong \sum_{e=1}^{n} \left\{ \sum_{j=1}^{m} \left[ \left( \sum_{i=1}^{l} N_i(\xi_j, \eta_j) \phi_i^e \right)^2 \left| J^e(\xi_j, \eta_j) \right| \right] w_j \right\} \quad (6)$$

where the only change from (1) is the squared basis function expansion. However, note that this squared expansion can be reinterpreted as a new field $\tilde{\phi}$ whose effective basis functions are defined by $\widetilde{N}_k(\xi,\eta) = N_i(\xi,\eta) \cdot N_j(\xi,\eta)$ and whose field values are $\tilde{\phi}_k^e = \phi_i^e \phi_j^e$ for all possible combinations of $i$ and $j$. Following this reinterpretation, the "stacking" method remains unchanged, only the number of effective basis functions and field values per element have increased.

Similarly, consider the diagonal stiffness matrix entries (again for brevity). Note that $\nabla\phi_p \cdot \nabla\phi_p = (\partial_x \phi_p)^2 + (\partial_y \phi_p)^2$ in Cartesian coordinates, where $\partial_x$ and $\partial_y$ denote the partial derivatives with respect to $x$ and $y$ respectively. It is sufficient to consider the first term $(\partial_x \phi_p)^2$ alone, to understand how the entries of the stiffness matrix can be computed using a modified "stacking" method approach. Using the chain rule to obtain expressions for the partial derivative of $N_i(\xi,\eta)$ with respect to $x$, and expanding $(\partial_x \phi_p)^2$ yields

$$S_x = \sum_{e=1}^{n} \left\{ \int_0^1 \int_0^{1-\xi} \sum_{i=1}^{l} \sum_{j=1}^{l} \frac{\phi_i^e \phi_j^e}{|\boldsymbol{J}^e|} \left[ \underbrace{\left( \frac{\partial y^e}{\partial \eta} \right)^2 \frac{\partial N_i}{\partial \xi} \frac{\partial N_j}{\partial \xi}}_{A_x'} \right. \right. \quad (7)$$
$$\left. \left. - \underbrace{2 \frac{\partial y^e}{\partial \xi} \frac{\partial y^e}{\partial \eta} \frac{\partial N_i}{\partial \xi} \frac{\partial N_j}{\partial \eta}}_{A_x''} + \underbrace{\left( \frac{\partial y^e}{\partial \xi} \right)^2 \frac{\partial N_i}{\partial \eta} \frac{\partial N_j}{\partial \eta}}_{A_x'''} \right] d\xi d\eta \right\},$$

where the $\partial_\xi y^e$ and $\partial_\eta y^e$ expressions depend on the curvilinear triangle geometry [1]. Consider $A_x'$ to formulate the "stacking" method for (7). $A_x''$ and $A_x'''$ are treated analogously.

In order to use the "stacking" method, all quantities in (7) that rely on element geometry must be characterized by a polynomial expression. Here, unlike in the previous two cases, the expression $(\partial_\eta y^e)^2 / |\boldsymbol{J}^e|$ is no longer polynomial in $\xi$ and $\eta$. For a planar, quadratic curvilinear, triangular geometry, the expression is a rational function of bivariate, complete, degree 2 polynomials. Equation (4) must be modified such that

$$I \cong \mathrm{diag}\left\{ \boldsymbol{N\Phi} \left[ (\boldsymbol{J}_{\eta^2} \boldsymbol{J}_b) \circ \widehat{(\boldsymbol{J}_a \boldsymbol{J}_b)} \right] \right\} \cdot \boldsymbol{w}, \quad (8)$$

where $\circ$ is the Hadamard product and $\widehat{\ }$ denotes the Hadamard inverse (i.e. $(Z \circ \hat{Z})_{ij} = 1$). Note that the matrix $\boldsymbol{J}_{\eta^2}$ has the same structure as $\boldsymbol{J}_a$, but with coefficients arising from the expression $(\partial_\eta y)^2$. Finally, re-indexing such that $\widetilde{N}_k(\xi,\eta) = \partial_\xi N_i(\xi,\eta) \cdot \partial_\xi N_j(\xi,\eta)$ with $\tilde{\phi}_k^e = \phi_i^e \phi_j^e$, the stiffness matrix can be computed using the "stacking" method.

## III. Results

Consider a long thin wire that carries a fixed current $I_0$ and lies on the $z$-axis, near a rectangular loop in the $yz$-plane. The magnetic flux through the loop can be calculated analytically, and is compared to the numerically computed flux through a curvilinear discretization of the rectangle. The smallest mesh considered is shown in Fig. 1. To test the efficiency of (8) – the "stacking" method applied to stiffness matrix entries – $\phi$ was specified so that the integrand $(\partial_y \phi)^2$ would represent the magnetic field intensity due to the current. Figure 2 compares the computational time for implementations of the "classical" integration and "stacking" methods, for increasing numbers of elements. The complete vectorization of the "stacking" algorithm across all elements results in a ten-fold reduction in cost. In the "classical" implementation, each element is treated separately and the algorithm cannot be fully vectorized. Both implementations converged to the same level of accuracy.

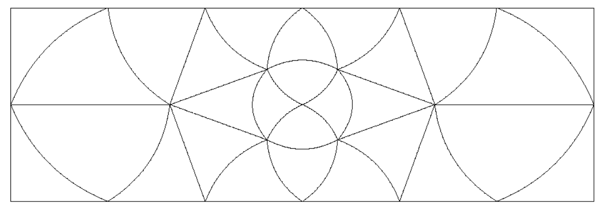

Fig. 1. FEM tile mesh consisting of 28 quadratic curvilinear triangles. Larger meshes were constructed by repeating the tile in both the $y$- and $z$-direction.

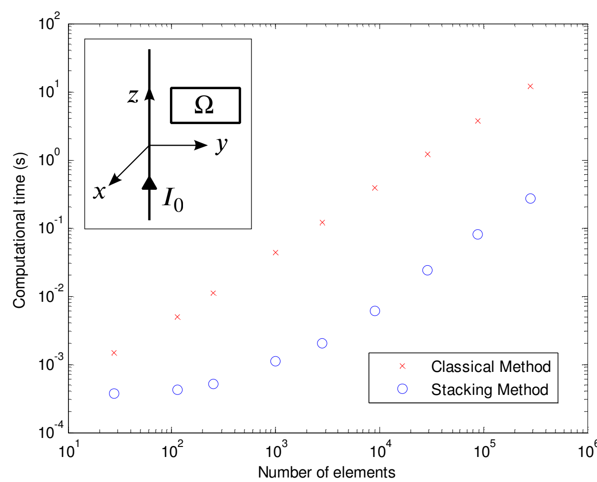


Fig. 2. Comparison of the efficiency computing $(\partial_y \phi)^2$ using both the "stacking" and "classical" methods, in terms of computational time.

## References

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