On Proper Orthogonal Decomposition for Electromagnetic Wave Problems

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***Abstract*—The proper orthogonal decomposition (POD) is one of the effective approaches for model order reduction. In this method, reduced models are constructed from relatively small number of the snapshots of original fields. This paper discusses application of POD to analysis of electromagnetic waves. The original wave problems are analyzed using finite element method where the perfect matched layer is employed to model absorbing boundaries. The reduced models in time domain are constructed from original solutions in either time-domain or frequency-domain. The accuracy and numerical stability of the present method are discussed.**

***Index Terms*—Wave propagation, model order reduction proper orthogonal decomposition, finite element method.**

I. Introduction

Model order reduction (MOR) has attracted attentions because it can effectively shorten the computational time of electromagnetic analysis. For example, the coupled analysis of circuits and antenna can easily be solved if the Maxwell equation for the antenna is transferred to ordinary differential equations with small number of unknowns. There are two approaches for MOR: one constructs the reduced system directly from the original system matrix while another generates it from the relatively small number of snapshots of the original fields. The former can effectively performed by the Pade-via-Lanczos method [1]. In fact, electromagnetic fields have successfully been analyzed using this method [2]. However it would be difficult to apply this method to wave problems with open boundaries as well as nonlinear problems. On the other hand, in the second approach called proper orthogonal decomposition (POD), the basis vectors of the reduced system are obtained by the principal component analysis in which the data matrix is constructed from the snapshots of the original fields. This method could be applied to a wider class of problems since POD can essentially be performed when the original problem can be solved. Magnetostatic and quasi-static problems have been solved using POD [3, 4]. However, it remains unclear if POD is valid for wave problems.

In this work, the reduced models of the wave problems are constructed on the basis of POD. The basis vectors of the reduced system are generated from original solutions in either time-domain or frequency-domain. The numerical stability, passivity, accuracy and effectiveness of this method will be discussed in detail.

II. Formulation

A. TD-FEM for wave problems

Let us consider wave propagation in the infinite finite domain, which is analyzed using finite element method (FEM). To model the absorbing boundary of the FE domain, we employ the perfect matched layer [5]. The governing equation in frequency domain is given by

$$\mathrm{rot}\,\nu\Lambda^{-1}(\omega)\cdot\mathrm{rot}\boldsymbol{E}-\omega^2\varepsilon\Lambda(\omega)\boldsymbol{E}=-\mathrm{j}\omega\boldsymbol{J}\,. \tag{1}$$

where

$$\Lambda(\omega)=\frac{s_y s_z}{s_x}\hat{x}\hat{x}+\frac{s_y s_z}{s_x}\hat{y}\hat{y}+\frac{s_y s_z}{s_x}\hat{z}\hat{z}\,, \tag{2}$$

$$s_\varsigma=1+\frac{\sigma_\varsigma}{\mathrm{j}\omega}\ (\varsigma=x,y,z)\,, \tag{3}$$

$\boldsymbol{E}$, ν, σ, ε, ω and $\boldsymbol{J}$ are electric field, magnetic resistivity, conductivity, permittivity, angular frequency and current density. To transform (1) to the time domain form, we apply the inverse Laplace transform to it to obtain

$$\mathrm{rot}\,\nu\mathcal{L}_1(t)\cdot\mathrm{rot}\boldsymbol{E}(t)-\varepsilon\mathcal{L}_2(t)\cdot\boldsymbol{E}(t)=-\frac{\partial\boldsymbol{J}}{\partial t}\,. \tag{4}$$

where

$$\mathcal{L}_1(t)=I+\frac{1}{\varepsilon}\mathrm{L}_1\mathrm{e}^{-\bar{\sigma}^{+}t/\varepsilon}u(t)*+\frac{1}{\varepsilon}\mathrm{L}_2\mathrm{e}^{-\sigma^{++}t/\varepsilon}u(t)*\,, \tag{5}$$

$$\mathcal{L}_2(t)=\mathrm{I}\frac{\partial^2}{\partial t^2}+\frac{1}{\varepsilon}\mathrm{J}\frac{\partial}{\partial t}+\frac{1}{\varepsilon^2}\mathrm{K}_1-\frac{1}{\varepsilon^3}\mathrm{K}_2\mathrm{e}^{-\bar{\sigma}t/\varepsilon}u(t)*\,, \tag{6}$$

and $u(t)$ and * denote the unit function and convolution operator and the definition of L_1, L_2, J, K_1 and K_2 is written by [6]. Applying the weighted residual method to (4), we obtain the FE equation given by

$$\mathrm{A}\frac{\partial^2\boldsymbol{e}}{\partial t^2}+\mathrm{B}\frac{\partial\boldsymbol{e}}{\partial t}+\mathrm{C}\boldsymbol{e}+\boldsymbol{h}(\boldsymbol{e})+\boldsymbol{g}(\boldsymbol{e})=\boldsymbol{b}\,, \tag{7}$$

where

$$A_{ij}=\int_V\varepsilon\boldsymbol{N}_i\cdot\boldsymbol{N}_j\mathrm{d}V\,, \tag{8}$$

$$B_{ij}=\int_V\boldsymbol{N}_i\cdot J\cdot\boldsymbol{N}_j\mathrm{d}V\,, \tag{9}$$

$$C_{ij} = \int_V \nu \mathrm{rot}\boldsymbol{N}_i \cdot \mathrm{rot}\boldsymbol{N}_j \mathrm{d}V + \int_V \frac{1}{\varepsilon} \boldsymbol{N}_i \cdot K_1 \cdot \boldsymbol{N}_j \mathrm{d}V, \tag{10}$$

$$h_i = -\int_V \frac{1}{\varepsilon^2} \boldsymbol{N}_i \cdot \sum_j (K_2 \cdot U_j \cdot \boldsymbol{N}_j) \mathrm{d}V, \tag{11}$$

$$g_i = \int_V \frac{\nu}{\varepsilon} \mathrm{rot}\boldsymbol{N}_i \cdot \sum_j \left((L_1 \cdot U_j^{+} + L_2 \cdot U_j^{++}) \cdot \mathrm{rot}\boldsymbol{N}_j \right) \mathrm{d}V, \tag{12}$$

and $\boldsymbol{N}_j$ is vector interpolation function and the definition of U_j, U_j^{+} and U_j^{++} can be found in [6]. We solve (7) by employing the Newmark Beta method for approximation of the time derivative to obtain the snapshots of the original solutions.

B. Proper Orthogonal Decomposition

In this paper, POD is applied to wave problems. In this method, from the snapshots of the original solution the basis vectors of the reduced system are generated via the principal component analysis. In this short paper, we consider the POD based on the time-domain snapshots $\boldsymbol{x}_k \in R^m$ (k=1, 2 ,3 …), from which the data matrix X is constructed as follows:

$$\mathrm{X} = [\boldsymbol{x}_1 - \boldsymbol{\mu} \quad \boldsymbol{x}_2 - \boldsymbol{\mu} \quad \cdots \quad \boldsymbol{x}_s - \boldsymbol{\mu}], \tag{13}$$

where s is the number of snapshots and $\boldsymbol{\mu}$ is the mean vector. To get the basis vectors, we apply the singular value decomposition to matrix X, which results in

$$\mathrm{X} = \mathrm{W}\Sigma\mathrm{V}^{\mathrm{t}} = \sigma_1 \boldsymbol{w}_1 \boldsymbol{v}_1^t + \sigma_2 \boldsymbol{w}_2 \boldsymbol{v}_2^t + \cdots + \sigma_s \boldsymbol{w}_s \boldsymbol{v}_s^t, \tag{14}$$

where $\boldsymbol{w}_i$ and $\boldsymbol{v}_i$ are the eigenvectors of the matrices XX^{t} and $\mathrm{X}^{\mathrm{t}}\mathrm{X}$, respectively and σ_i, i=1,2,…,s are the singular values which correspond to the square root of the eigenvalues of XX^{t}. The original unknown vector $\boldsymbol{x}$ is approximately expressed by the linear combination of the reduced vector $\boldsymbol{y} \in R^s$ as follows:

$$\boldsymbol{x} = \mathrm{W}\boldsymbol{y}. \tag{15}$$

Now we can transform (7), which is here expressed by $\mathrm{K}\boldsymbol{x}=\boldsymbol{b}$ for simplicity, to the reduced form as follows:

$$\mathrm{W}^{\mathrm{t}}\mathrm{K}\mathrm{W}\boldsymbol{y} = \mathrm{W}^{\mathrm{t}}\boldsymbol{b}. \tag{16}$$

III. NUMERICAL RESULT

We solve the numerical model shown in Fig. 1 using the present method, where a sinusoidal current flows around the brick shaped magnetic material. FE model has 15600 nodes, 13718 elements and 44840 edges. The driving current and frequency f is 1AT and 150MHz. Time step Δt is set to $1/250f$. The permeability μ_r in the magnetic material is 10. The results are shown in Fig. 2 where (a) and (b) show the distribution of magnetic flux density computed by the conventional FEM and the present method with s=100. We find in Fig.2 that both magnetic fields are almost identical. This suggest that the present method can provide accurate reduced system, which would be useful for fast simulation of antennas loaded by electric circuits where circuit parameters are changed for their design.

IV. CONCLUSION

In this paper, we have presented the model order reduction based on POD for the wave problem. Although the present is accurate in generation of the reduced model of the numerical example, we still need careful discussion on its performance. We will apply the method which determines the adequate number of snapshots automatically [4] to wave problems. Moreover, POD based on the snapshots in frequency domain will be discussed in the full paper.

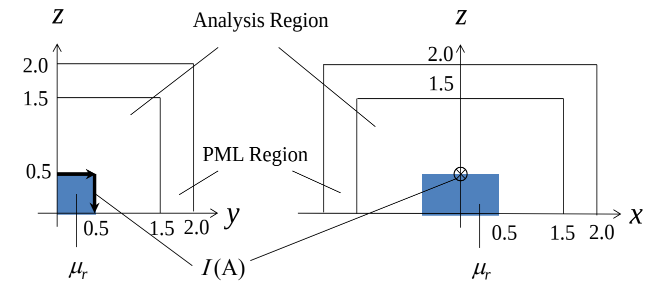
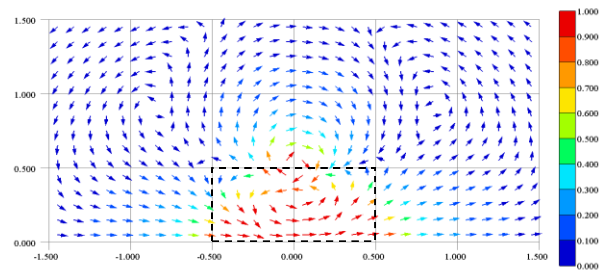
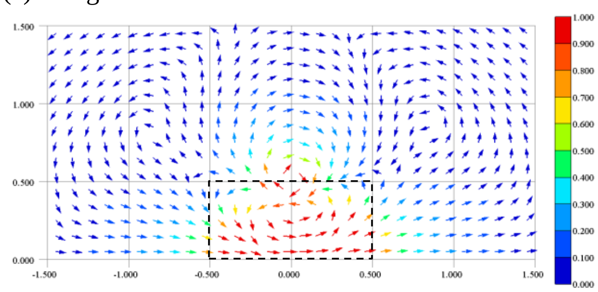


Fig. 1. Numerical model



(a) Original solution



(b) Solution obtained by present method with s=100.
Fig. 2.Distribution of magnetic flux density

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