The FEM-SRBCI Method for the Solution of Scalar Scattering Problems

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Abstract— **In this paper the authors propose a modified version of the hybrid FEM-RBCI method to solve scalar scattering problems, such as 2D electromagnetic and 3D acoustic ones. In the modified method, called FEM-SRBCI, the integration surface coincides with the truncation one, so that the integral equation becomes singular.**

Index Terms—**Scattering, Computational electromagnetics, Finite element methods.**

I. INTRODUCTION

In order to solve electromagnetic scattering problems, the authors have devised a hybrid method, called FEM-RBCI (Finite Element Method – Robin Boundary Condition Iteration) [1-5]. FEM-RBCI couples a differential equation, which governs the interior problem, with an integral one which makes use of the free-space Green function and expresses the unknown boundary condition on the fictitious truncation boundary. In [1] the authors have shown that only the use of suitable Robin boundary conditions avoid internal resonances, whatever the frequency of the incident wave.

In FEM-RBCI the truncation boundary includes the integration surface, which on its turn includes the scatterer. In this way no singularity arises in the integral equation. This advantage is balanced by the drawback of the presence of finite elements in between the truncation and integration surfaces.

This paper presents a modified version of the FEM-RBCI method in order to eliminate this drawback. In this version the truncation and integration surfaces are coincident, so that the integral equation becomes singular; from this the name FEM-SRBCI (Singular RBCI).

II. THE FEM-SRBCI METHOD FOR SCATTERING

Consider a system of conducting and/or dielectric objects, infinitely extended in the z-direction, surrounded by free space. The system is radiated by a given time-harmonic electromagnetic wave E_{inc}, E-polarized along the z-axis. A scattered field E_{scat} is excited extending to infinity. The total field E (outside the scatterers $E=E_{inc}+E_{scat}$) satisfies the 2D scalar Helmholtz equation:

$$\nabla \cdot (\mu_r^{-1}\nabla E) + k_0^2 \varepsilon_r E = 0 \qquad (1)$$

where, μ_r and ε_r are the relative magnetic permeability and electric permittivity, respectively, and k_0 is the free-space wavenumber. Homogeneous Dirichlet conditions hold on the perfect conductor surfaces Γ_C, if any. Moreover E_{scat} must satisfy the Sommerfeld radiation condition at infinity.

In order to compute the field inside the penetrable objects and in the proximity of the perfect conductors (PEC), they are enclosed in a fictitious boundary, Γ_F, where a Robin condition is initially imposed:

$$\Re E = \frac{\partial E}{\partial n} + jk_0 E = \psi \qquad (2)$$

the normal derivative being calculated in the outward direction. By discretizing the domain D, delimited by Γ_C and Γ_F, by means of Lagrangian finite elements and applying the Galerkin method, the following set of algebraic equations is obtained:

$$\mathbf{AE} = \mathbf{B\Psi} \qquad (3)$$

where $\mathbf{E}$ and $\mathbf{\Psi}$ are the vectors of the nodal values of E and ψ, respectively, $\mathbf{A}$ is an FEM global matrix, and $\mathbf{B}$ is a rectangular sparse matrix of geometrical coefficients.

The field E in a point $\mathbf{r}$ on Γ_F can be expressed by means of the Green formula:

$$\frac{\gamma}{2\pi}E(\mathbf{r}) = E_{inc}(\mathbf{r}) - \int_{\Gamma_F}\left\{G(\mathbf{r},\mathbf{r}')\frac{\partial E(\mathbf{r}')}{\partial n'} - E(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\right\}ds' \qquad (4)$$

and hence:

$$\frac{\gamma}{2\pi}\psi(\mathbf{r}) = \Re E_{inc}(\mathbf{r}) - \int_{\Gamma_F}\left\{\Re G(\mathbf{r},\mathbf{r}')\frac{\partial E(\mathbf{r}')}{\partial n'} - E(\mathbf{r}')\frac{\partial \Re G(\mathbf{r},\mathbf{r}')}{\partial n'}\right\}ds' \qquad (5)$$

where γ is the angle of the external domain at point $\mathbf{r}$, n' is the outward normal unit vector at point $\mathbf{r}'$ and G is the 2-D free-space Green's function:

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4}jH_0^{(2)}\left(k_0|\mathbf{r}-\mathbf{r}'|\right) \qquad (6)$$

$H_0^{(2)}$ being the is the zero-order Hankel function of the second kind.

In the FEM approximation, relation (5) is rewritten as:

$$\mathbf{C\Psi} = \mathbf{\Psi}_{inc} + \mathbf{HE} \qquad (7)$$

where $\mathbf{C}$ is a diagonal matrix containing the $\alpha/2\pi$ coefficients and $\mathbf{H}$ is a rectangular matrix in which null columns appear for the nodes not belonging to the elements having a side lying on Γ_F. For the computation of the coefficients in matrix $\mathbf{H}$, the following integrals are computed:

$$h'_{ik} = \int_{S_k}\frac{\partial\alpha_i(\mathbf{r}')}{\partial n'}\Re G(\mathbf{r},\mathbf{r}')ds' \qquad (8)$$

$$h''_{ik} = \int_{S_k}\alpha_i(\mathbf{r}')\frac{\partial\Re G(\mathbf{r},\mathbf{r}')}{\partial n'}ds' \qquad (9)$$

where α_i is the shape function of the i-th node P_i and S_k is the k-th side of the finite elements lying on Γ_F. If node P_i does not

belong to side S_k, Gauss integration is used; otherwise analytical (or numerical) integration formulas are used [6].

Equations (3) and (7) constitute a linear algebraic system which can be efficiently solved with a block Gauss-Seidel iteration scheme: i) at the beginning the vector Ψ is guessed arbitrarily (a good choice is $\Psi=\mathbf{C}^{-1}\Psi_{inc}$); ii) equation (3) is solved for $\mathbf{E}$; iii) another guess for Ψ is obtained by means of (7); iv) if convergence is reached the procedure stops; otherwise it goes back to ii).

Computational efficiency is obtained by fully exploiting the following points in implementation: a) matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{H}$ do not change during the iterations, so they are computed only once, at the beginning of the procedure; b) equation (3) may be solved by means of standard solvers, which exploit the matrix $\mathbf{A}$ sparsity and symmetry; c) by suitably placing Γ_F around the PEC scatterers a small extension of free-space has to be meshed; d) the end-iteration test is conveniently restricted to the truncation boundary Γ_F.

III. FEM-SRBCI FOR 3D PROBLEMS

The FEM-SRBCI method described in Sect. II can be easily extended to 3-D problems. Of course the 3-D version does not apply to electromagnetic phenomena, since in this case both the equation and the Robin boundary condition change in a vectorial manner [4]. However there exist several kinds of 3-D physical phenomena, such as acoustic ones, which are governed by the scalar Helmholtz equation and to which the FEM-SRBCI method is fully applicable.

In the 3-D version of the procedure the main changes are concerned with the Green's function to be used, which in this case is given by:

$$G(\mathbf{r},\mathbf{r}') = -\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad (10)$$

Of course Γ_F is now a closed surface, which is seen as constituted by finite element faces, that is, triangles [7] or quadrangles for tetrahedral or brick elements, respectively.

IV. NUMERICAL EXAMPLES

Two numerical examples are given. The first concerns a plane wave lighting a perfectly-conducting circular cylinder, of radius $R_C=2.25\lambda$, coated with a lossy dielectric ($\varepsilon_r=1.5-j0.8$, $\mu_r=2-j$) of thickness $a=0.25\lambda$. A circular Γ_F was selected coinciding with the scatterer surface. For symmetry reasons only half of the domain D was discretized with a mesh of 1300 2nd-order triangles (130 subdivisions along φ, 5 along r). Assuming an end-iteration tolerance of 1%, the procedure converges in 5 iterations. The numerical solution was compared with the analytical one: the relative error was 0.5%. Similar results were also obtained in the H-polarized case.

The second example regards a multiple scatterer composed of a dielectric ($\varepsilon_r=1-j0.2$, $\mu_r=2-j$) circular cylinder (of radius $R=0.8\lambda$) and a perfectly-conducting triangular cylinder. An E-polarized plane wave comes from the left. The fictitious boundary Γ_F was selected as constituted of two closed curves: one coincides with the surface of the circular cylinder, and the other is homologous to the triangular
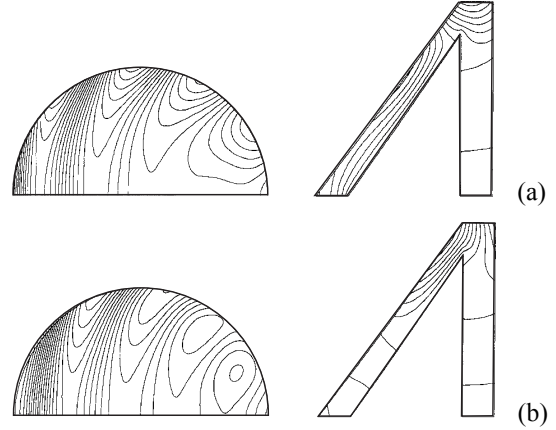

Fig. 1 – Contours of the real (a) and imaginary (b) parts of E.

cylinder. A mesh of 408 2nd-order triangles was adopted to discretize half of the system. The procedure converged in 8 iterations. Figs. 1.a and 1.b show the contours of the real and imaginary parts of the electric field, respectively. The solution was compared with another one obtained by FEM-RBCI, in which a single fictitious boundary enclosed the two scatterers. a relative error of 0.4% was observed.

The computations were performed by means of ELFIN, a large FEM code developed by the authors for electromagnetic CAD research [8]. More details and example will be provided in the full paper.

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