Cutoff Wavenumbers of Circular Metallic Waveguides with Eccentricity

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***Abstract*— Cutoff wavenumbers k_{nm} are determined analytically for a circular metallic waveguide with eccentricity. Separation of variables technique is used for the solution. For small eccentricities *kd*, where d is the distance between the axes of the cylinders the cosine and the sine laws are used, in order to satisfy the boundary conditions at the surface of the outer cylinder. Keeping terms up to the order $(kd)^2$ exact, analytical expressions of the form $k_{nm}(d)= k_{nm}(0)[1+ g_{nm}(k_{nm}d)^2+O(k_{nm}d)^4]$ are obtained for the cutoff wavenumbers of the waveguides, where $k_{nm}(0)$ corresponds to the coaxial geometry, with d=0. Both TM and TE modes are considered. Numerical results are given for all types of modes and for various values of the parameters. The method used here is an alternative of the one using the translational addition theorem, in the case of small eccentricities.**

***Index Terms*— Electromagnetic propagation, Closed-form solution.**

I. Introduction

Determination of the cutoff wavenumbers for TM and TE modes of circular waveguides with eccentricity is a problem that has been of interest to many researchers. Various techniques [1-7] have been used in order to obtain numerical results. In this paper the cutoff wavenumbers for eccentric circular perfectly conducting metallic waveguides are obtained using a shape perturbation method. The radii of the outer and the inner cylinder are a and b, respectively, while the distance between their axes is d (Fig. 1).

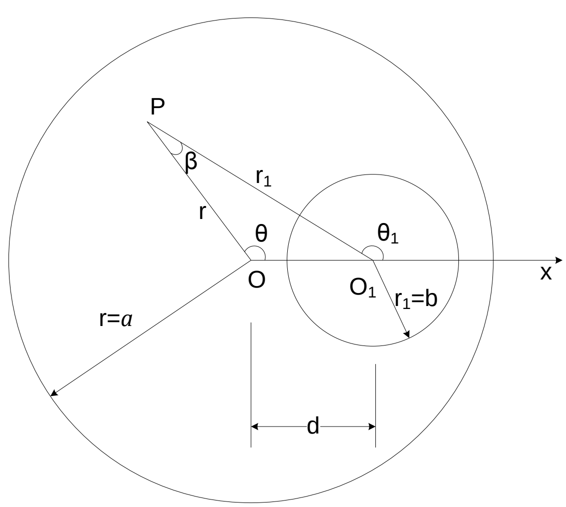

Fig. 1. Eccentric circular waveguide cross-section

II. Solution of the Problem

For the transverse magnetic (TM) modes the electromagnetic field at the general point P(r_1,θ_1,z), where r_1,θ_1 are the polar coordinates with respect to O_1x, can be expressed in terms of circular cylindrical wavefunctions. Once the boundary condition $E(b,\theta_1,z)=0$ is satisfied, the longitudinal component takes the form

$$E_z = e^{-j\beta z}\sum_{n=0}^{\infty}\left[J_n(kr_1)-Y_n(kr_1)J_n(kb)/Y_n(kb)\right] \cdot\left[A_n\cos(n\theta_1)+B_n\sin(n\theta_1)\right], \tag{1}$$

where the time dependence $\exp(j\omega t)$ is suppressed throughout. J_n and Y_n are the cylindrical Bessel functions of the first and second kind, respectively, and $k=(\omega^2\varepsilon\mu-\beta)^{1/2}$ is the wavenumber, with ε the permittivity and μ the permeability of the material inside the waveguide. In order to satisfy the boundary condition at $r=a$ the cosine and the sine laws are used to express r_1 and θ_1 in terms of r and θ. For small values of d, keeping terms up to the order d^2, one gets the expressions (see also [8])

$$r_1 = r - d\cos\theta + \frac{\sin^2\theta}{2r}d^2 + O\left(d^3\right) \tag{2}$$

and

$$\theta_1 = \theta + \frac{d}{r}\sin\theta + \frac{1}{2}\left(\frac{d}{r}\right)^2\sin(2\theta) + O\left(d^3\right). \tag{3}$$

Satisfying the boundary condition $E(a,\theta_1,z)=0$, using the above expressions, and, finally, the orthogonal properties of trigonometric functions, we obtain two infinite homogenous sets of linear equations. Keeping terms up to the order $(kd)^2$, these sets have the following forms:

$$\begin{aligned} &a_{\nu,\nu-2}A_{\nu-2} + a_{\nu,\nu-1}A_{\nu-1} + a_{\nu\nu}A_\nu \\ &\qquad + a_{\nu,\nu+1}A_{\nu+1} + a_{\nu,\nu+2}A_{\nu+2} = 0, \quad \nu \ge 0 \\ &b_{\nu,\nu-2}B_{\nu-2} + b_{\nu,\nu-1}B_{\nu-1} + b_{\nu\nu}B_\nu \\ &\qquad + b_{\nu,\nu+1}B_{\nu+1} + b_{\nu,\nu+2}B_{\nu+2} = 0, \quad \nu \ge 1. \end{aligned} \tag{4}$$

According to the even or odd angular symmetry of E_z, with respect to the x axis, two independent types of TM modes exist, even (ETM) and odd (OTM) ones. For nontrivial solutions the determinants of the two infinite sets should vanish, leading to determinantal equations from which the cutoff wavenumbers may be determined. For small eccentricities ($kd << 1$) one is led to an exact evaluation of second order terms, in *kd*, for the elements of the former infinite determinants and, finally, for the determinants themselves. Following the steps of [1] it can be shown that for small *d*'s each cutoff wavenumber $k_{nm}(d)$ has values near to the corresponding one $k_{nm}(0)$ of the coaxial waveguide. Thus, the cutoff wavenumbers are obtained in the following form:

$$k_{nm}(d) = k_{nm}(0)\left[1 + g_{nm}(k_{nm}d)^2 + O(k_{nm}d)^4\right], \qquad (5)$$

with g_{nm} having exact, analytical, closed-form expressions, independent of the eccentricity *d*. A similar procedure is followed for both types of TE modes, even (ETE) and odd (OTE) ones. Numerical results are obtained for all types of modes and for various values of the parameters and are compared to those given in [3] and [4]. The comparison proves the validity of our method even for values of $kd > 1$.

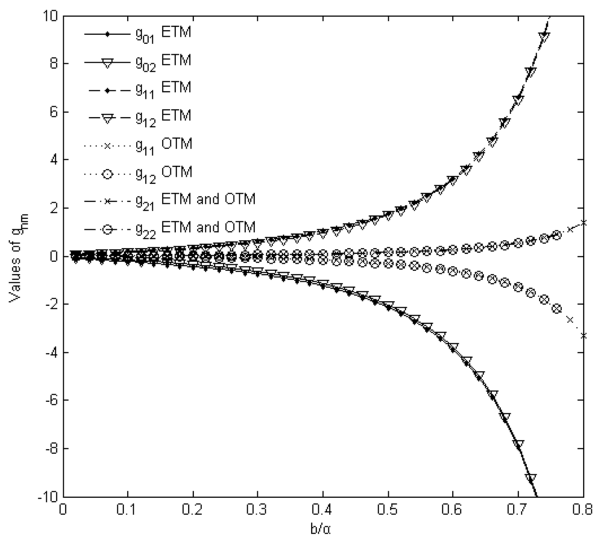

Fig. 2.Values of g_{nm} for the TM modes

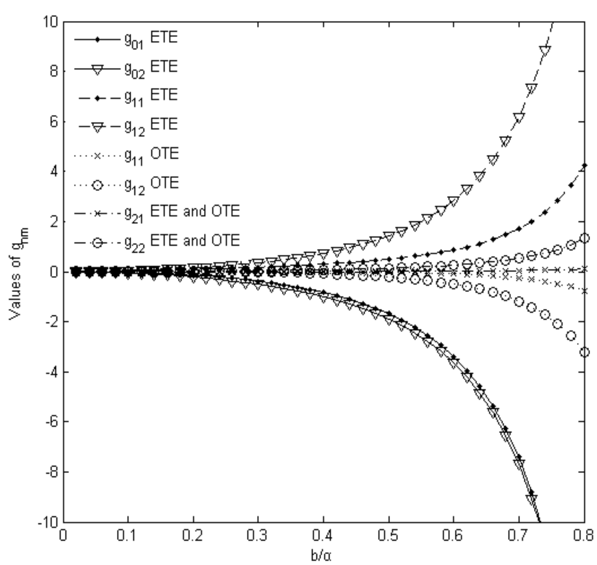

Fig. 3. Values of g_{nm} for the TE modes

In Figs. 2 and 3 the g_{nm} coefficients for some of the lower order modes are plotted versus *b/a* for TM and TE modes, respectively. It can be seen that their absolute values become too small for small values of *b/a*, something that follows also from further available results and permits the use of very large values for *d/b* in these cases.

Although many numerical solutions for the same problem already exist, the main advantage of our method is that it is valid for any small value of *kd* and for all modes. Once the expansion coefficients g_{nm} are known, the cutoff wavenumbers can be immediately evaluated quickly by "back-of-the-envelope" calculations, while all numerical techniques require repetition of the evaluation for each *d* and for each mode. Since the terms omitted here are of the order of $(kd)^4$ and higher the restriction $kd << 1$ is not as severe as it may initially appear. This is supported by the comparison of our results against already existing ones. The proposed method is an alternate method of the one using the translational addition theorem [1]

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