Eddy Currents Computation by an Integral Equation Method Using Facet Elements

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Abstract— A formulation of the integral method for calculating time dependent electromagnetic fields and the eddy currents in conducting objects is considered in this paper. The current density vector is approximated by facet vector functions. Independent variables are associated with the co-tree branches of the graph built at the basis of triangular or tetrahedral mesh covering the space filled with conducting material. The algebraic equations for the unknown current density values are formed for the closed loops corresponding to the co-tree branches.

Index Terms—magnetic field, eddy currents, integral equation, finite element method, facet elements.

I. Formulation of the Problem

Integral methods are used successfully for modeling time dependent electromagnetic fields in conducting objects [1, 2]. Typically these formulations exploit different electromagnetic potentials or their combinations. In this paper we consider a formulation of such problem with the current density vector being the main variable.

Let us consider a system of equations governing the electromagnetic field inside conducting media:

$$\begin{aligned} &rot\vec{H} = \vec{J}\,, \\ &div\vec{B} = 0\,, \\ &rot\vec{E} = -\frac{d\vec{B}}{dt}\,. \end{aligned} \tag{1}$$

Here we suppose that the field characteristics change in time slowly enough, and the displacement currents may be neglected. To complete formulation of the problem it is necessary to add equation for the current density continuity

$$div\vec{J} = 0\,, \tag{2}$$

and the standard constitutive relations for the conducting non-magnetic media:

$$\vec{J} = \gamma\vec{E}\,, \qquad \vec{B} = \mu_0\vec{H}\,.$$

The field intensity at any point of the considered space may be expressed as a superposition of the external field $\vec{H}_m$ and the field $\vec{H}_c$ induced by the eddy currents circulating in the conducting object:

$$\vec{H} = \vec{H}_c + \vec{H}_m$$

Both components of the magnetic field may be calculated using Biot-Savart law:

$$\vec{H}_c(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}dV'\,. \tag{3}$$

Such presentation of the field intensity automatically fulfills the first and the second equations of the system (1) for the currents satisfying (2). Combining (3) and (1) we can derive an integro-differential equation:

$$rot\vec{J} = -\gamma\mu_0\frac{d}{dt}\left[\frac{1}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}dV' + \vec{H}_m\right]. \tag{4}$$

Together with the relation (2) and appropriate boundary conditions on the conductor surface the last equation defines a unique solution for the current density vector.

II. Approximation of the Integro-Differential Equation

To solve numerically the equation (4) we split space filled with the conductor into a set of triangular or tetrahedral elements depending on the space dimension. The unknown current density distribution is approximated by a superposition of the vector finite functions associated with the facets of the generated mesh:

$$\vec{J}(r) = \sum_{i=1}^{F} J_i\vec{\varphi}_i(\vec{r}) \tag{5}$$

with $\vec{\varphi}(\vec{r})$ being the facet functions and J_i the components of the current density normal to the facets. Such choice of the current density approximation provides the continuity of the normal components of this vector and gives a simple way to set appropriate boundary conditions.

Evidently some of the values J_i are not independent because they should satisfy restriction (2). So a proper choice of the independent variables and corresponding functions should be performed. For this purpose we build a graph corresponding to the tetrahedral (or triangular in the case of a 2D problem) mesh. The nodes of the graph are associated with the central points of each element. Each branch of the graph connects the centers of neighboring elements and so crosses one facet. Example of such graph corresponding to 2-dimensional object is shown in Fig. 1. The full graph is separated into the main tree and the co-tree. Every section of the graph (closed line crossing only one tree branch) forms a closed surface (or loop in 2D) and so may be used to set a constitutive relation, equivalent to the equation (2):

$$\sum_{i=1}^{T} w_i l_i J_i = 0,$$

l_i is the area of the corresponding facet (length of the edge in 2D), $w_i = \pm 1$ is a multiplier depending on the direction of the current density with respect to normal to the facet vector. Consequently only the variables corresponding to the co-tree branches may be regarded as independent.

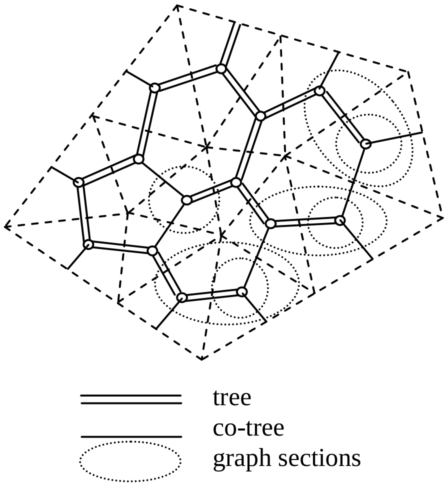

Fig.1. A graph corresponding to the triangular mesh.

Similar formulation of this problem for the case of thin conducting plates when the surface charges induced by the eddy currents may be neglected was described in [3]. In the general case the algebraic equations for the unknown current density values may be formed for the closed loops consisting of the graph branches. Such loop should include only one co-tree branch, while all others should belong to the main tree. Integration of the equation (4) along such lines gives:

$$\oint_l \vec{J}(\vec{r}) \cdot d\vec{l} + \frac{\mu\gamma}{4\pi}\frac{d}{dt}\oint_l d\vec{l} \int \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|} dV' = -\frac{\mu\gamma}{\mu_0}\oint_l \frac{d\vec{A}(\vec{r})}{dt} d\vec{l},$$

where $\vec{A}(\vec{r})$ is the magnetic vector potential induced by external sources. To derive the last equation we transformed integrals over the area to the integrals over the bounding closed lines using the Stokes' theorem. Evidently the number of the equations is equal to the number of unknowns and all these equations are linearly independent. Presentation of the current density vector $\vec{J}(\vec{r})$ in a form of (5) gives a system of algebraic equations with the unknown values J_i.

III. Test Problem

To verify the proposed algorithm we considered a long hollow conducting cylinder in the uniform external magnetic field with the harmonic time dependence. The direction of the field intensity is parallel to the cylinder axis. The inner and outer radii are equal to 0.5 and 1.0 m. This problem was chosen for analysis because the corresponding analytical solution may be easily derived. The penetration depth of the electromagnetic field into the conducting material in this example is $\lambda = 0.1$ m. The results of the eddy current computation are shown in Fig.2 – Fig.3.

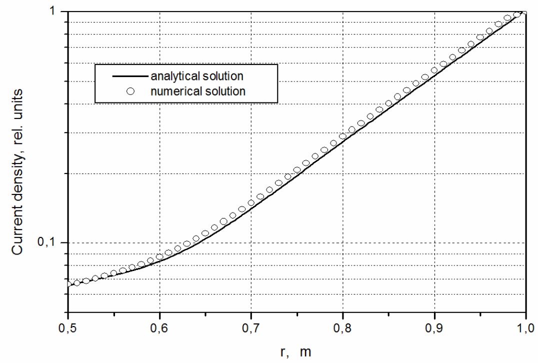

Fig.2. Dependence of the current density on the radius for λ=0.1 m.

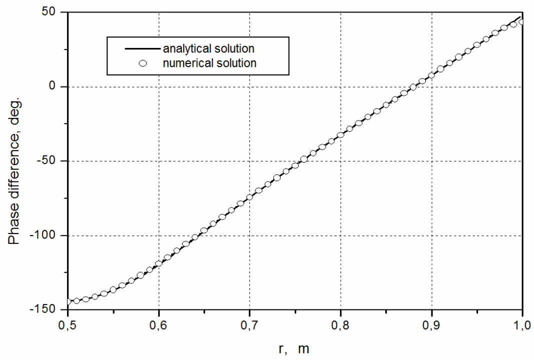

Fig.3. Dependence of the current density phase on the radius for λ=0.1 m.

These distributions demonstrate a good agreement between the numerically and analytically derived data.

IV. Conclusions

In this paper we describe a universal integral method for the eddy current calculation. Approximation of the unknown current density vector is done on the basis of the facet functions. Proper choice of the independent unknowns provides the solution of the problem which satisfies one of the Maxwell equations (2) by default. The algebraic equations approximating the integral equation (4) are formed for the closed loops corresponding to the co-tree branches. The proposed algorithm is verified by calculating eddy currents in a hollow conducting cylinder.

V. References

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