Accurate Post-Processing of Magnetic Field Gradients from Low-Order Finite-Element Solutions

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***Abstract*— Accurate results for the field gradient in a Stern-Gerlach magnet are obtained by calculating a local analytic solution using a finite-element solution as boundary conditions. In contrast to standard post-processing based on numerical differentiation, the improved technique preserves the convergence order for the discretisation error.**

***Index Terms* – Finite element methods, magnetostatics, convergence of numerical methods, accelerator magnets, magnetic separation.**

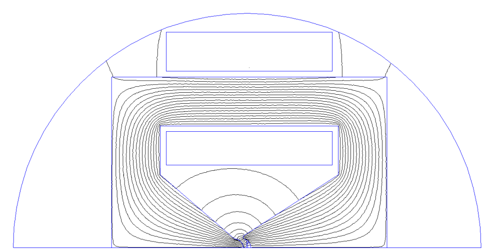

Fig. 1. Magnetic flux lines in a Rabi-type magnet (half model).

I. Introduction

The Stern-Gerlach experiment splits a particle beam according to their quantum spin [1]. For that purpose, the beam is brought through a region with a large field gradient. The original setup consists of two parallel wires, which allows to express the field gradient by an analytic formula. Today, a Rabi-type magnet with an iron yoke, a concave and a convex pole shoe is used [2]–[4] (Fig. 1).

The design of a Rabi-type deflection magnet is supported by 2D nonlinear magnetostatic finite-element (FE) field simulation. A particular numerical challenge arises when calculating the field gradients during post-processing. Two successive numerical differentiations are needed to derive the gradient of the magnetic flux density $\mathbf{B}$ from the FE solution for the magnetic vector potential $\mathbf{A}$. When linear FE shape functions are used, $\mathbf{A}$ converges like $\mathcal{O}(h^2)$ where h stands for a characteristic mesh size, whereas $\mathbf{B}$ converges like $\mathcal{O}(h)$ and the field gradient may not converge at all [5] (Fig. 3). It is obvious that this fact endangers the application of FE simulation for designing a deflection magnet.

The problem can be alleviated by taking higher other FE shape functions [6], inserting a boundary-element subdomain [7] or applying spectral elements in the magnet aperture [8], [9]. These methods all have their virtues, but cause a considerable increase of the computational cost of the field calculation. Here, we choose to keep a standard nonlinear 2D FE solver and develop a new and dedicated post-processing tool calculating magnetic field gradients while preserving maximal convergence order. Similar approaches have been reported for post-processing torques [10] and post-processing boundary-element solutions [11].

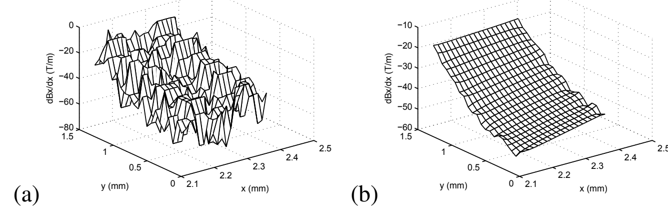

Fig. 2. Gradient in the x-direction of the x-component of the magnetic flux density in the magnet aperture; comparison between (a) the solution obtained by standard post-processing and (b) by the improved technique.

II. Standard Post-Processing for Field Gradients

The magnetic flux density $(B_x, B_y) = (\frac{\mathrm{d}A_z}{\mathrm{d}y}, -\frac{\mathrm{d}A_z}{\mathrm{d}x})$ is found by differentiating the FE shape functions. When piecewise linear shape functions are applied for A_z, (B_x, B_y) is piecewise constant per element. A direct further differentiation for the field gradient would lead to a zero solution. A straightforward remediation of this problem consists of interpolating the element-wise magnetic flux density onto the nodes, i.e., $\mathbf{B}_i = \sum_{k\in\mathcal{N}_i} S_k\mathbf{B}_k / \sum_{k\in\mathcal{N}_i} S_k$ where elements k are selected from the neighbourhood $\mathcal{N}_i$ of node i and S_k denotes the area of element k. The nodal solution is seen as a piecewise linear solution in terms of the same shape functions and can be further differentiated. When higher-order FE shape functions are used, the element-to-node interpolation is not needed. From Fig. 2a, it is obvious that this procedure leads to poor results. Therefore, an improved post-processing technique is indispensable when field gradients need to be determined with the highest possible accuracy.

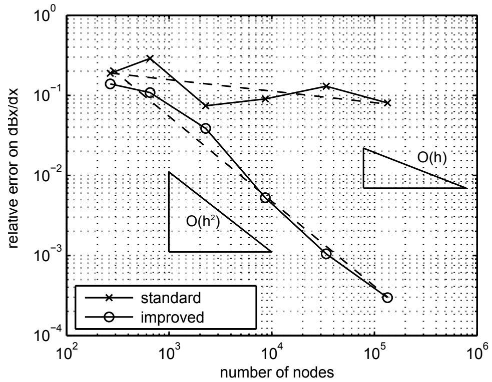

Fig. 3. Convergence of the discretisation error for the gradient of the x-component of the magnetic flux density in the x-direction. Comparison of the standard post-processing approach by numerical differentiation of the polynomial solution with the improved approach based on a local solution of the Laplace equation.

III. Improved Post-Processing for Field Gradients

By a standard 2D nonlinear magnetostatic FE solver, the spatial distribution of the z-component $A_z(x,y)$ of the magnetic vector potential is calculated. On a rectangular domain of size Δx-by-Δy in the magnet aperture, a local analytical solution of the governing Laplace equation $\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}=0$ is constructed using the FE solution as boundary conditions:

$$\begin{aligned} A_z^{(\mathrm{lpl})}(x,y) &= \alpha+\beta x+\gamma y+\delta xy \\ &+ \sum_{p=1}^{\infty}\sin\left(\xi_p x\right)\left[a_p\cosh\left(\xi_p y\right)+b_p\sinh\left(\xi_p y\right)\right] \\ &+ \sum_{q=1}^{\infty}\sin\left(\eta_q y\right)\left[c_q\cosh\left(\eta_q x\right)+d_q\sinh\left(\eta_q x\right)\right] \end{aligned} \quad (1)$$

where $\xi_p=\frac{p\pi}{\Delta x}$ and $\eta=\frac{q\pi}{\Delta y}$. The coefficients α, β, γ and δ follow from the values of A_z at the corner nodes. The coefficients a_p, b_p, c_q and d_q are found by 1D Fast Fourier Transforms [12] of the remaining potential distribution at the sides of the rectangular boundary. Field gradients can be obtained from (1) by double differentiation without loss of convergence order, e.g., for $\frac{\mathrm{d}B_x}{\mathrm{d}x}$,

$$\begin{aligned} \frac{\mathrm{d}B_x^{(\mathrm{lpl})}}{\mathrm{d}x}(x,y) &= \delta \\ &+ \sum_{p=1}^{\infty}\xi_p^2\cos\left(\xi_p x\right)\left[a_p\sinh\left(\xi_p y\right)+b_p\cosh\left(\xi_p y\right)\right] \\ &+ \sum_{q=1}^{\infty}\eta_q^2\cos\left(\eta_q y\right)\left[c_q\sinh\left(\eta_q x\right)+d_q\cosh\left(\eta_q x\right)\right] \end{aligned} \quad (2)$$

It may make sense to truncate the series in (2) after a relatively small amount of terms to diminish numerical instabilities because of the amplifying factors ξ_p^2 and η_q^2. The detour around the analytical solution avoids numerical differentiation [13]. The additional computation work may adversely influence the accuracy. Nevertheless, the convergence order for $\frac{\mathrm{d}B_x}{\mathrm{d}x}$ remains the same as for the original FE solution for A_z.

IV. Validation, Application and Conclusions

The improved post-processing technique is validated for the two-wire configuration for which a comparison with an analytical formula is possible. It is obvious from Fig. 2b that the local post-processing technique improves the smoothness of the field gradient substantially. In Fig. 3, the convergence of the discretisation error is compared. The field gradient obtained by standard post-processing converges with an order between $\mathcal{O}(1)$ and $\mathcal{O}(h)$, whereas the improved technique attains the convergence order $\mathcal{O}(h^2)$, which is same order as for the magnetic vector potential and predicted by theory.

The improved technique is applied to Rabi-type Stern-Gerlach magnets (Fig. 1). The FE solutions are obtained by FEMM [14]. The post-processing is based on the above discussed techniques. The increased accuracy and reliability allows to embed the FE simulation in parameter loops and optimisation steps used during the magnet design. Generalisations to the 3D case, local post-processing regions with other shapes and configurations with particular symmetries are straightforward.

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