New methods based on matched coordinates for the computation of quasi-static fields induced in a layered conductor with a rough surface and a continuous depth profile of conductivity

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Abstract—**This paper deals with the computation of quasi-static fields induced in a conductor by a 3D Eddy Current (EC) probe. The shape of the slab is locally distorted and constituted by several homogeneous or non-homogeneous layers with non-parallel interfaces. The approach is based on writing Maxwell's equations in a curvilinear system, leading to a simple analytical expression of boundary conditions. In homogeneous layers, the fields are expanded as sums of eigenfunctions. In the case of non-homogeneous layers, a pseudo-spectral method is introduced and combined with modal solutions to solve Maxwell's equations. Some numerical experiments validate this innovative approach, resulting in a fast model, able to tackle a complex configuration never solved in the context of EC Nondestructive Testing.**

Index Terms—**Computational Electromagnetics, Eddy currents, Nondestructive testing, Surface roughness, Electrical Conductivity Measurement**

I. INTRODUCTION

The analysis of the residual stresses inside a material after any subsurface treatment is of great interest in order to improve the time life of critical parts. By using ECNDT techniques, the subsurface residual stress distribution may be related with the depth profile of the electrical conductivity of the material. The characterization of such material requires to develop fast numerical models for solving direct and inverse problems. Some analytical models already exist for investigating planar structures, providing the depth profile of the conductivity fits some specific known mathematical function in order to obtain an analytical solution to the Helmoltz's equation [1], [2]. The problem of any arbitrary continuous depth profile may be addressed by different numerical methods. Indeed, it is possible to approach the continuous depth profile by a constant piecewise function, the non-homogeneous medium being replaced by a stack of homogeneous layers. Besides, an alternative method based on the Taylor's expansions of the profile has been recently proposed [3].

In this paper, we propose an innovative solution which combines both any arbitrary continuous depth profile of the conductivity and a local complex shape of the structure under test as depicted on Figure 1. The so-called Curvilinear-

Method, based on the covariant form of the Maxwell's equations and usually used in the high frequency domain (optics for instance) has been recently adapted to the computation of quasi-static fields induced in 2D [4] or 2D1/2 [5] homogeneous conductors with a rough surface. We propose here to couple this method with a pseudo-spectral method [6] in order to be able to take into account the continuous variations of the conductivity in any non homogeneous layer. For this purpose, the covariant Maxwell's equations expressed in the Fourier domain are solved by a combination of modal solutions for homogeneous mediums (air and substrate for example...) with a numerical pseudo-spectral solution in the non homogeneous layers. Some numerical validations are given in order to show the validity and the efficiency of this innovative method.
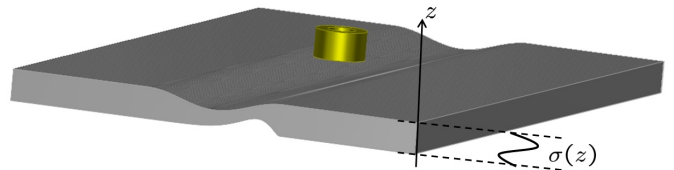

Figure 1: A complex ECNDT configuration combining two consecutive rough interfaces and a continuous depth profile of the electrical conductivity. This slab may be excited by any air-core 3D EC probe.

II. FORMALISM

A. *Change of coordinate system*

The modal method previously obtained was based on a translative change of coordinate system:

$$\begin{cases} x^1 = x \\ x^2 = y \\ x^3_m = z - a_m(x), \end{cases} \tag{1}$$

where m denotes the mth interface described by an analytical expression $z = a_m(x)$. By fitting a new coordinate system corresponding to each interface, the distorted shape of the slab is included in the metric tensor $G_m = (g_{m,ij})$ and its inverse:

$G_m^{-1} = (g_m^{ij})$. Boundary conditions are thus written at each mth interface with $x_m^3 = constant$.

B. Covariant Maxwell equations

In these non-orthogonal systems, the covariant form of Maxwell's equations can be written for each layer under a compact form [7]:

$$\begin{bmatrix} -k_c \sqrt{g_m} g_m^{ij} & \nabla \times \mathbf{I} \\ \nabla \times \mathbf{I} & -k_c \sqrt{g_m} g_m^{ij} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ Z\mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad (2)$$

where k_c is the wave number in the medium ($k_c^2 \simeq j\omega\mu\sigma$ in a conductive medium), $g_m = det(G_m)$, Z is the characteristic impedance of the medium, $\mathbf{I}$ stands for the identity operator.

C. Modal expansions in homogeneous layers

In homogeneous layers, the electromagnetic fields can be expressed in the Fourier domain [8] (the spatial frequencies are chosen along the axis x^1 and x^2), in terms of two scalar potentials Γ and Π satisfying a single eigenvalue system [5]. Each potential and its derivative along the x^3 axis can be expanded in a linear combination of eigenfunctions ψ_m and ϕ_m depending on the interface $a_m(x)$:

$$\begin{bmatrix} \widehat{\Gamma}_m \\ \partial_3 \widehat{\Gamma}_m \end{bmatrix} = \begin{bmatrix} \psi_{m,q}^+ & \psi_{m,q}^- \\ \phi_{m,q}^+ & \phi_{m,q}^- \end{bmatrix} \begin{bmatrix} e^{j\lambda_{m,q}^+ x_m^3} & \mathbf{0} \\ \mathbf{0} & e^{j\lambda_{m,q}^- x_m^3} \end{bmatrix} \begin{bmatrix} \Gamma_{m,q}^+ \\ \Gamma_{m,q}^- \end{bmatrix} \quad (3)$$

The coefficients $\Gamma_{m,q}^+$ and $\Gamma_{m,q}^-$ are determined by writing boundary conditions. This modal approach has been validated for a stratified medium considering that the depth profile of conductivity is a constant piecewise function. For a continuous profile, another complementary approach is proposed.

D. Pseudo-spectral approach in non-homogeneous layers

In non homogeneous layers, the eigenvalue system is no longer available and the covariant form of Maxwell's equations expressed in Fourier domain are thus discretized along x^3 (which is the direction of variation of the conductivity) by using a Chebyshev differentiation matrix [6]. A set of $N + 1$ Chebyshev points are distributed as follows:

$$\xi_{m,j} = \frac{x_m^3 - x_{m-1}^3}{2} cos\left(\frac{j\pi}{N}\right) + \frac{x_m^3 + x_{m-1}^3}{2}, \quad j = 0, 1, \ldots, N$$

where x_m^3 and x_{m-1}^3 stand for the depth of the two consecutive interfaces and $x_{m-1}^3 \le \xi_{m,j} \le x_m^3$. Moreover, a linear transition function $h(\xi)$ is added in order to link the two different shapes as depicted in Figure 2. $h(x_m^3) = 0$ at the upper interface whereas $h(x_{m-1}^3) = 1$ at the lower one.
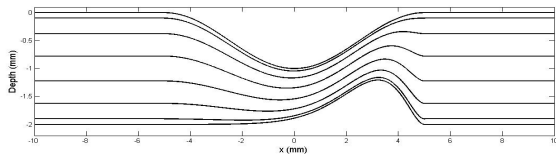

Figure 2: Discretization of the x^3 axis in one of the non homogeneous layers: vizualisation of the effect of the transition function $h(\xi)$ for $N = 7$.

By introducing the Chebyshev differentiation matrix on the $N-1$ intermediate interfaces, ones obtains a numerical operator corresponding to Maxwell's equations in non homogeneous layers. Two other boundary conditions are finally added in order to link the modal expansions in the farthest layers (in air) to the numerical solution in non homogeneous layers.

III. NUMERICAL VALIDATION RESULTS

The impedance Z of the 3D EC probe is compared to other numerical results provided by the full modal method considering that the depth profile conductivity may be approximated by a constant piecewise function (see Figure 3).
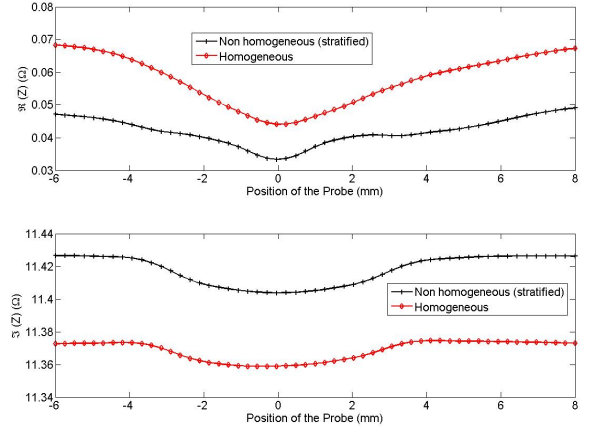

Figure 3: Real and imaginary part of the total impedance of a 3D air-core EC probe during a scan over the inhomogeneous piece ($\sigma(z) = \sigma_0[1 + arctan(z)]$, with $\sigma_0 = 1$ MS/m) approached by a multilayered structure (N = 9) and the homogeneous piece ($\sigma = \sigma_0$) presenting the same geometry. The operating frequency is 10 kHz. The structure is displayed in Fig 2.

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