Computing Eddy Currents in Thin Shells of Arbitrary Topology by Mesh Analysis

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Abstract—**A novel 3-D integral formulation for solving eddy currents in thin conducting structures with complex topology is presented. As in discrete approaches field problem variables are cochains, dual to chains defined on a pair of interlocked cell complexes. So the discretized problem can be cast in a circuit-like manner. The proposed method is formulated in terms of mesh currents on a triangulated surface in order to reduce computing requirements compared to edge element formulations. The basic idea is to use homology generators of the triangulated surface to compute the additional cochains needed by mesh analysis.**

Index Terms—**Eddy currents, Topology, Electromagnetic shielding, Computational geometry, Circuit analysis.**

I. INTRODUCTION

Computing eddy currents in thin conducting structures embedded in unbounded air domains is computationally demanding when 3-D finite element formulations with volume elements are used. In that case a huge amount of elements with bad aspect ratio is required for discretizing conducting regions. Moreover, meshing of air domains further increases the number of degrees of freedom. Both of these aspects lead to poor numerical performance and accuracy.

A viable solution is to use integral formulations which do not require the discretization of the air region [1]. This reduces time needed for pre-processing, e.g. model setup and meshing, and, in turn, computing requirements during solution. Another advantage during post-processing is that fields are computed from equivalent sources only in limited regions, instead of the whole computational domain as with finite element method.

Integral formulations based on the Cell Method (CM) have been proposed for analyzing EM shielding problems at very low frequency [2][3]. Hybrid formulations based on CM and BEM have been developed for analyzing quasi-magnetostatic problems [4][5]. These formulations are not suitable to compute models with complex topology, e.g. shells with holes or bulk domains with cavities. In [6], [7] and [8] surface integral equation methods have been proposed for solving thin shell problems with multiply connected domains. Among these, the numerical strategy shown in [6] is particularly advantageous since unknowns are related to interior nodes and holes.

A 3-D integral formulation for computing eddy currents in thin conducting structures of arbitrary topology—in particular multiply connected—is presented. With mesh currents as problem unknowns as in Circuit Theory the number of degrees of freedom for solving eddy currents is minimal with benefits in terms of computing requirements.

II. COHOMOLOGY BASIS

The well-posedness of eddy current problems is related to a proper definition of the algebraic structure of solution space, which is related in turn to the topology of the computational domain. Topological bases for the integral formulation developed in the next section are described below.

Thin conducting structures can be approximated as equivalent surfaces when the eddy current density is approximately uniform across the thickness, i.e. when the skin depth is greater than the shell thickness. Let $\Gamma \subset \mathbb{R}^3$ be a closed surface, i.e. compact and without boundary, representing the thin structure and Γ_h its triangulation. Let K be the simplicial complex related to Γ_h and $\hat{K}$ its (barycentric) dual complex, constructed by joining the centroids of $\hat{K}$. A k-chain is a formal sum of k-simplices of K, i.e. $c = \sum_i a_i \sigma_i$, $a_i \in \mathbb{C}$. In the same way, dual k-chains $\hat{\sigma}$ can be built on $\hat{K}$. The boundary operator ∂ from k-chains to $(k-1)$–chains allows building the (simplicial) chain complex of K, from which simplicial homology spaces $H_k(\Gamma; \mathbb{C})$ are obtained [9]. These are useful to characterize surface topology with linear algebra for numerical computing.

Simplicial cohomology spaces $H^k(\Gamma; \mathbb{C})$ are the complex vector spaces dual to $H_k(\Gamma; \mathbb{C})$. Representatives of homology classes in $H^k(\Gamma; \mathbb{C})$ are k-cochains, i.e. linear forms mapping k-chains to complex numbers. Poincaré duality for a closed surface states that $H^1(\Gamma; \mathbb{C})$ is isomorphic to $H_1(\Gamma; \mathbb{C})$ so that cohomology basis can be built from homology basis. From De Rham's theorem $H^1(\Gamma; \mathbb{C})$ is also isomorphic to De Rham cohomology space $H^1_{\mathrm{dR}}(\Gamma)$, i.e. the quotient space of closed k-forms $Z^k_{\mathrm{dR}}(\Gamma)$ modulo exact k-forms $B^k_{\mathrm{dR}}(\Gamma)$. Therefore, any k-cochain can be expressed as linear map $c \mapsto \int_c \omega$ for some k-form ω. Viceversa, ω defines a k-cochain on its turn.

The eddy current density on Γ (in the quasi magneto-static limit) can be regarded as a closed 1-form, i.e. $\mathrm{d}\omega = 0$. Therefore, ω is an element of $Z^1_{\mathrm{dR}}(\Gamma)$ and the corresponding coset is $[\omega] = \omega + B^1_{\mathrm{dR}}(\Gamma)$. If $[\omega_i]$, $i = 1 \ldots n$, where n is the first Betti number, is the (finite) basis of $H^1_{\mathrm{dR}}(\Gamma)$—the cohomology basis—, then $[\omega] = \sum_{i=1}^n \alpha_i [\omega_i]$ for some complex coefficients α_i. Equating cosets yields $\omega = \omega_0 + \sum_{i=1}^n \alpha_i \omega_i$, where $\omega_0 \in B^1_{\mathrm{dR}}(\Gamma)$. If Γ is of arbitrary topology, ω cannot be expressed simply as the differential of a stream function f. By letting $\omega_0 = \mathrm{d}f$, the eddy current density becomes

$$\omega = \mathrm{d}f + \sum_{i=1}^n \alpha_i \omega_i \tag{1}$$

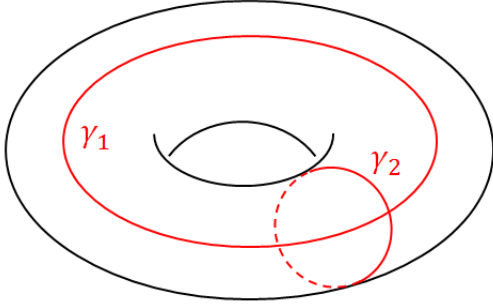Figure 1: Loops γ_1, γ_2 are associated to representatives of a vector basis (equivalence classes) of space $H_1(T; \mathbb{C})$.

Numerical procedures for finding cohomology generators of electric vector potential formulations in bulk domains have been proposed in [10][11]. It can be observed that, by combining De Rham and Poicaré isomorphisms, a correspondence between the homology and cohomology bases is established and generators ω_i can be obtained. In this work the algorithm proposed in [12] for building a shortest homology basis has been used. Its basic advantage is to minimize generator supports and, thus, number of coupling terms in linear systems.

III. INTEGRAL FORMULATION

According to the so-called *discrete approaches* such as the Cell Method, field problems can be formulated in terms of 1-cochains and dual 1-cochains. Let be $C^1(K)$ the complex space of 1-cochains on K and $C^1(\hat{K})$ the complex space of dual 1-cochains on $\hat{K}$. By taking vector bases of these spaces and from (1), dual 1-cochain $\hat{c} \mapsto \int_{\hat{c}} \omega$ can be expressed into a discrete form with coefficient vectors and matrices, as

$$\mathbf{i} = \mathbf{C}^\mathrm{T} \mathbf{i_o} + \mathbf{Q}^\mathrm{T} \mathbf{i_t} \qquad (2)$$

where $\mathbf{i}$ is the array of eddy currents on 1-simplices of $\hat{K}$ related to ω, $\mathbf{i_o}$ the array of mesh currents related to f, $\mathbf{i_t}$ is the array of topological mesh currents related to ω_i (of size n), $\mathbf{C}$ is the edge-to-cell incidence matrix and $\mathbf{Q}$ is the edge-to-homology loop incidence matrix. Fig. 1 shows an example of homology loops for a torus T: Each loop is associated to a representative of a basis vector (equivalence class) of $H_1(T; \mathbb{C})$.

Because of $\mathrm{d}\omega = 0$, the combinatorial expression (2) has to identically fulfill the div-free condition at the algebraic level

$$\mathbf{G}^\mathrm{T} \mathbf{i} = \mathbf{0} \qquad (3)$$

where $\mathbf{G}$ is the node-to-edge incidence matrix. The kernel of $\mathbf{G}^\mathrm{T}$ is spanned by the columns of mesh incidence matrix $\mathbf{C_m^T}$, which is built by assembling column-wise incidence matrices $\mathbf{C}$ and $\mathbf{Q}$ in (2). In such a way the eddy current problem is well-posed also in the case of non-trivial domains.

The topological equation (3) for the cell complex $\hat{K}$ must be complemented with a topological equation for the simplicial complex, involving magnetic fluxes $\mathbf{b}$ and induced emfs $\mathbf{e}$, as

$$\mathbf{C} \mathbf{e} + j w \mathbf{b} = \mathbf{0} \qquad (4)$$

where w is the angular frequency of magnetic field sources. With (2) additional field problem unknowns, i.e. mesh currents $\mathbf{i_t}$, have been added in order to account for domains of arbitrary topology. Additional constraints are thus required to determine mesh currents, which are given again by Faraday's law (4).

Constitutive equations map dual 1-cochains to 1-cochains. These are constructed by approximating 1-forms with suitable local basis functions. In the present work, div-conforming piece-wise uniform basis functions, proposed in [6], are used to approximate ω locally. The electrical and magnetic constitutive relationships are $\mathbf{e} = \mathbf{R}\mathbf{i}$ and $\mathbf{a} = \mathbf{L}\mathbf{i}$, where $\mathbf{a}$ is the array related to the magnetic vector potential 1-form, and $\mathbf{R}$ and $\mathbf{L}$ are the resistance and inductance matrix, respectively. Magnetic fluxes in (4) can be expressed as a function of magnetic vector potentials as $\mathbf{b} = \mathbf{C}\mathbf{a}$.

The relationship (2) can be rewritten in compact form as $\mathbf{i} = \mathbf{C_m^T}\mathbf{i_m}$, where $\mathbf{i_m} = (\mathbf{i_0}\,\mathbf{i_t})^\mathrm{T}$ is the array of mesh currents. Including additional constraints for topological mesh currents, (4) becomes $\mathbf{C_m}\mathbf{e} + jw\mathbf{b} = \mathbf{0}$ and $\mathbf{b} = \mathbf{C_m}\mathbf{a}$. By inserting electric and magnetic constitutive relations, the following linear matrix system is finally obtained

$$\left(\mathbf{C_m^T}\,\mathbf{Z}\,\mathbf{C_m}\right)\mathbf{i_m} = -jw\,\mathbf{b_s} \qquad (5)$$

where $\mathbf{Z} = \mathbf{R} + jw\mathbf{L}$ is the impedance matrix, typical of mesh analysis. The array of currents can be computed after solving mesh analysis. A thorough discussion of the proposed integral formulation and of numerical results are presented in the paper.

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