

Fast Calculation of Magnetic Fields Produced by Rectangular Cross Section, Arc-Shaped Conductors

Alessio Capelluto[°], Mario Nervi[#], Paolo Molfino[#]

[°]ASG Superconductors S.p.A.

Corso F. M. Perrone 73r, I-16152 Genova, Italy

[#]Department of Electrical, Electronics, Telecommunications Engineering and Naval Architecture,

University of Genova, Via Opera Pia 11a, I-16145 - Genova, Italy

capelluto.alessio@as-g.it, mario.nervi@unige.it, paolo.molfino@unige.it

Abstract — This paper presents a novel procedure for the fast numerical integration of magnetic field produced by arc-shaped conductors, characterized by rectangular cross section. Available procedures are based on the analytical integration of Biot-Savart's law but, most of them, exploit the analytical integration along one coordinate, and then perform a two dimensional numerical integration. In the proposed procedure, the analytical integration was performed along two coordinates, obtaining a one dimensional integrand (thus avoiding the use of Elliptic Integrals), very easy to process using a state-of-the-art numerical quadrature library. The result is very satisfactory in terms of high speed and precision, particularly on conductor surface, and when its cross dimensions are very uneven.

I. INTRODUCTION

Magnetic field generated by current flowing into conductors is calculated through the integration of Biot-Savart's law, that sadly cannot be evaluated in closed form for arbitrary conductor shapes. For bar-shaped conductors close formulas are available. In the other cases, direct numerical integration, though possible, is computationally intensive and subject to numerical problems due to the intrinsically singular structure of the integrand. Most researchers then developed methods characterized by a first analytical integration along one coordinate, and then a two dimensional numerical integration [1-2], usually involving elliptic integrals [3-7]. For this reasons, a novel procedure was developed that, integrating analytically along two coordinates, leads to a one-dimensional integrand. Such integrand (though characterized by some singularities) can easily be integrated by a numerical quadrature procedure. The proposed approach follows a logical procedure similar to EFFI [8], but appears to be even simpler and more compact.

II. THE ANALYTICAL DEVELOPMENTS

Starting from Biot-Savart's law (1), it is well known that, for bar-shaped conductors, an analytical solution providing the field in every point does exist. The problems arise with arc-shaped conductor, indispensable to model complex coils.

$$\vec{B} = \frac{\mu_0}{4\pi} \int_l \iint_S \frac{\vec{J} \times (\vec{x} - \vec{x}_0)}{|\vec{x} - \vec{x}_0|^3} \quad (1)$$

Assuming a cylindrical local coordinate system R, φ, Z , defined on the arc axis, the arc can be expressed as:

$$r_1 \leq R \leq r_2 \quad (2)$$

$$z_1 \leq Z \leq z_2 \quad (3)$$

$$\varphi_1 \leq \varphi \leq \varphi_2 \quad (4)$$

The first analytical integration is then performed along Z ; the second along R ; the procedure is relatively straightforward for B_x and B_y , i.e. the components laying on planes parallel to the current. On the contrary, the analytical integration is more involved for B_z , which usually is also the most important component. Naming field point coordinates as ρ, θ, z we get:

$$zmt = z_2 - z \quad (5)$$

$$zmd = z_1 - z \quad (6)$$

$$\rho_2 = \rho^2 \quad (7)$$

$$T_1 = \rho \cdot \cos(\varphi - \theta) \quad (8)$$

$$R_A = \rho \cdot \sin(\varphi - \theta) \quad (9)$$

$$R_{A2} = R_A^2 \quad (10)$$

$$R_1 = \sqrt{r_2^2 + \rho_2 - 2T_1r_2 + zmt^2} \quad (11)$$

$$R_2 = \sqrt{r_2^2 + \rho_2 - 2T_1r_2 + zmd^2} \quad (12)$$

$$R_3 = \sqrt{r_1^2 + \rho_2 - 2T_1r_1 + zmt^2} \quad (13)$$

$$R_4 = \sqrt{r_1^2 + \rho_2 - 2T_1r_1 + zmd^2} \quad (14)$$

$$A = R_1 - r_2 + T_1 \quad (15)$$

$$B = R_2 - r_2 + T_1 \quad (16)$$

$$C = R_3 - r_1 + T_1 \quad (17)$$

$$D = R_4 - r_1 + T_1 \quad (18)$$

$$n_1 = (zmt + A)^2 + R_{A2} \quad (19)$$

$$n_2 = (zmd - B)^2 + R_{A2} \quad (20)$$

$$n_3 = (zmt - C)^2 + R_{A2} \quad (21)$$

$$n_4 = (zmd + D)^2 + R_{A2} \quad (22)$$

$$d_1 = (zmt - A)^2 + R_{A2} \quad (23)$$

$$d_2 = (zmd + B)^2 + R_{A2} \quad (24)$$

$$d_3 = (zmt + C)^2 + R_{A2} \quad (25)$$

$$d_4 = (zmd - D)^2 + R_{A2} \quad (26)$$

$$M = n_1 n_2 n_3 n_4 / d_1 d_2 d_3 d_4 \quad (27)$$

$$n_{f1} = \text{atan2}(-R_A, zmt + A) \quad (28)$$

$$n_{f2} = \text{atan2}(R_A, zmd - B) \quad (29)$$

$$n_{f3} = \text{atan2}(R_A, zmt - C) \quad (30)$$

$$n_{f4} = \text{atan2}(-R_A, zmd + D) \quad (31)$$

$$d_{f1} = \text{atan2}(R_A, zmt - A) \quad (32)$$

$$d_{f2} = \text{atan2}(-R_A, zmd + B) \quad (33)$$

$$d_{f3} = \text{atan2}(-R_A, zmt + C) \quad (34)$$

$$d_{f4} = \text{atan2}(R_A, zmd - D) \quad (35)$$

$$f = (n_{f1} + n_{f2} + n_{f3} + n_{f4}) - (d_{f1} + d_{f2} + d_{f3} + d_{f4}) \quad (36)$$

$$L_z = zmt \ln\left(\frac{A}{C}\right) + zmd \ln\left(\frac{D}{B}\right) \quad (37)$$

$$L_{xy} = R_1 - R_2 - R_3 + R_4 + T_1 \ln\left(\frac{(2R_1-A)(2R_4-D)}{(2R_3-C)(2R_2-B)}\right) \quad (38)$$

$$B_x = \frac{J\mu_0}{4\pi} \int_{\varphi_1}^{\varphi_2} L_{xy} \cdot \cos(\varphi) d\varphi \quad (39)$$

$$B_y = -\frac{J\mu_0}{4\pi} \int_{\varphi_1}^{\varphi_2} L_{xy} \cdot \sin(\varphi) d\varphi \quad (40)$$

$$B_z = -\frac{J\mu_0}{4\pi} \int_{\varphi_1}^{\varphi_2} (L_z + \frac{1}{2}T_1 \cdot \ln(M) + f \cdot R_A) d\varphi \quad (41)$$

where “atan2” has the same meaning of Fortran language.

To calculate the components of the induction, three 1D integrals must be evaluated (39-41). We decided to use the numerical quadrature package TOMS691 [9], in double precision version. Such library is based upon Gaussian quadrature integration; it is therefore extremely useful as it can naturally treat integrand singularities (due to the property that Gauss integration never samples the integrand on interval boundaries). The chosen routine was the DQAGP, which provides adaptive integration of singular integrands. The singularities do arise from the above reported eqs. in the cases:

$$\varphi = \theta \pm \arccos\left(\frac{\sqrt{\rho^2 + (z_k - z)^2}}{\rho}\right) + N\pi, k \in \{1,2\}, N \in \{0,1\} \quad (42)$$

$$\varphi = \theta \pm \arccos\left(\frac{\rho^2 + r_k^2}{2\rho r_k}\right), k \in \{1,2\} \quad (43)$$

It is apparent that singularities may only occur if the field point lies on conductor surface, for $\theta = \varphi$ and $\theta = \varphi + \pi$. Also for some field points belonging to the z axis the method cannot provide a numeric solution; in this case the well-known analytical formulas are used to compute the correct value of the magnetic induction.

III. RESULTS

Tests were performed on the required design specifications of the algorithm, to assess the speed, the accuracy of the results, and the stability of the numerical result when the aspect ratio of coils cross section tends to be very unfavourable. All test cases were run on a PC equipped with a 3.16 GHz CPU.

A. Radial path across an arc coil

This case is characterized by a radial path completely contained into the conductor, going from the internal face up to the external one. Usually the surface of conductors is the most critical area, especially for superconductors. The arc is $\frac{1}{4}$ of a solenoid. In Fig. 1 the comparison between the result from the proposed algorithm and commercial software is reported. The two curves are perfectly overlapping.

B. Tangential path along the internal face of an arc coil

This case was developed to test the behavior of the integration procedure along the angular variable. To solve the numerical problems arising in the singular azimuthal coordinate points (eqs. (42) and (43)), they had to be identified and passed to the integration routine. The testing proved this strategy successful. The arc is $\frac{1}{4}$ of a solenoid.

C. Semicircular path along the axis of a solenoidal coil

This test case was needed to verify the quality of the result on a sphere, centered on the axis of a solenoid, but with centre not coincident with the solenoid centre. Due to the symmetry of the problem the field was just computed on a plane with fixed angular coordinate, leading to a semicircular path.

In Tab. I the column labeled “Avg. samplings” reports the total number of samplings of the integrand, divided by the number of field points (10^4 in our tests). With standard algorithms, C.P. times were about fifty times larger than ours, even though it is hard to make exact time measurements: in commercial codes the integration is deeply embedded into the code, and source code is not available to place time markers.

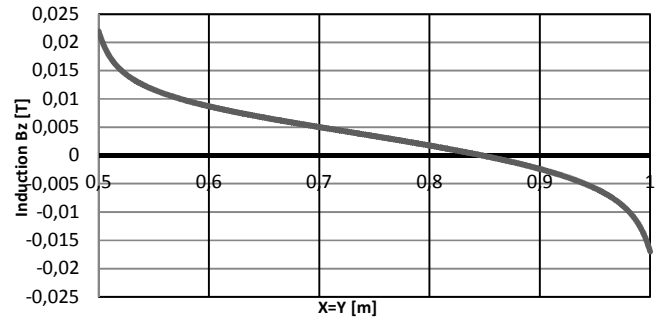


Fig. 1. Magnetic induction B_z for test case A.

TABLE I
C.P. TIMES FOR 10^4 FIELD POINTS, SAMPLINGS AND ACCURACY

Case	Avg. samplings	Abs. Accuracy [T]	Total C.P. Time [s]
A	228	1.0e-04	1.02
A	548	1.0e-09	2.87
B	412	1.0e-04	2.31
B	915	1.0e-09	4.59
C	21	1.0e-04	0.19
C	21	1.0e-09	0.19

IV. REFERENCES

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