Modeling and Finite Element Simulation of the Wilson–Wilson Experiment

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***Abstract*—The classical Wilson–Wilson experiment in 1913 about the electromagnetic field in a rotating non-conducting cylinder is studied, based on Finite Element analysis. Electric and magnetic fields are coupled through motional terms in the constitutive relations. This particular kind of coupling is usually not considered in existing numerical models. Effects due to the finite axial extension of the cylinder can be conveniently assessed with the help of the Finite Element simulation. This should be useful for the correct interpretation of the experimental data.**

***Keywords*—Electromagnetic coupling, numerical simulation.**

I. Introduction

The classical Wilson–Wilson experiment [1] has re-gained attention, both from the experimental [2] and conceptual point of view [3]–[6], since there was still some controversy about the agreement of the experiment with relativity. The experiment consists of a magnetic nonconducting hollow rotating cylinder which is immersed in the external magnetic field of a solenoid, Fig. 1. According to the relativistic theory of moving media [7] a potential difference occurs between the inner and outer surfaces of the cylinder, which bear metallic coatings.

In general, motion might exhibit itself through (i) a time-dependent geometry, (ii) motional terms in the interface conditions [8], or (iii) the constitutive relations. In usual low-frequency electromagnetic modeling, (iii) applies to Ohm's law, $\boldsymbol{J} = \kappa(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$, where $\boldsymbol{J}$ is the electric current density, κ the conductivity, $\boldsymbol{E}$ the electric field, $\boldsymbol{v}$ the velocity, and $\boldsymbol{B}$ the magnetic flux density. Interestingly, the Wilson–Wilson experiment cannot be described in terms of such a quasistatic model, because there is no separation of inductive and capacitive effects. Moreover, due to lack of bulk conductivity, Ohm's law is irrelevant, and the motion enters the constitutive relations of the electromagnetic fields. This renders the situation interesting from the modeling and simulation point of view. Numerical simulations also allow for assessing effects due to finite axial extension of the cylinder and imperfect external field.

II. Modeling

Away from the coil the system is described by time-independent homogeneous Maxwell equations

$$\begin{aligned} \mathrm{curl}\boldsymbol{H} &= 0\,, \quad \mathrm{curl}\boldsymbol{E} = 0\,, \\ \mathrm{div}\boldsymbol{B} &= 0\,, \quad \mathrm{div}\boldsymbol{D} = 0\,, \end{aligned} \tag{1}$$

where $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{D}$ the electric flux density. If the considered domain is simply connected, we can introduce electric and magnetic scalar potentials by $\boldsymbol{E} = -\mathrm{grad}\varphi$ and $\boldsymbol{H} = -\mathrm{grad}\psi$, respectively.

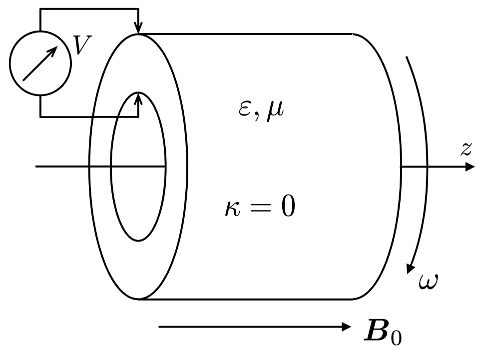

Figure 1. The Wilson–Wilson experiment. A magnetic ($\mu > \mu_0$) nonconducting ($\kappa = 0$) hollow cylinder is immersed in the external magnetic field $\boldsymbol{B}_0$ of a solenoid. It rotates at constant angular velocity ω around its axis. A potential difference V occurs between the inner and outer surfaces of the cylinder, which bear metallic coatings.

The general form of the interface conditions at a moving interface is (see Table 1 in [8])

$$\begin{aligned} \boldsymbol{n}\times[\boldsymbol{H}] + (\boldsymbol{n}\cdot\boldsymbol{v})[\boldsymbol{D}] &= \mathbf{k}\,, & \boldsymbol{n}\times[\boldsymbol{E}] - (\boldsymbol{n}\cdot\boldsymbol{v})[\boldsymbol{B}] &= 0\,, \\ \boldsymbol{n}\cdot[\boldsymbol{B}] &= 0\,, & \boldsymbol{n}\cdot[\boldsymbol{D}] &= \sigma\,, \end{aligned} \tag{2}$$

where $[\,\cdot\,]$ denotes the jump in the direction of the unit normal vector $\boldsymbol{n}$, $\mathbf{k}$ is the surface current and σ the surface charge density. In the Wilson–Wilson experiment, the geometry is stationary, which amounts to $\boldsymbol{n}\cdot\boldsymbol{v} = 0$ on the boundary of the hollow cylinder. The interface conditions reduce to their usual form. That is, the motion is neither "felt" (i) through a time-dependent geometry nor (ii) through the interface conditions.

The *Minkowski relations* established in 1910 are [7], [9]

$$\begin{aligned} \boldsymbol{D} + 1/c_0^2\,\boldsymbol{v}\times\boldsymbol{H} &= \varepsilon(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\,, \\ \boldsymbol{B} - 1/c_0^2\,\boldsymbol{v}\times\boldsymbol{E} &= \mu(\boldsymbol{H} - \boldsymbol{v}\times\boldsymbol{D})\,, \end{aligned} \tag{3}$$

where $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ is the velocity of light in empty space, ε is the permittivity and μ the permeability. We are using the constitutive relations in the form $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E},\boldsymbol{H},\boldsymbol{v})$, $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{E},\boldsymbol{H},\boldsymbol{v})$, in their low-velocity approximation

$$\begin{aligned} \boldsymbol{D} &= \varepsilon\boldsymbol{E} + \frac{\lambda}{c^2}\boldsymbol{v}\times\boldsymbol{H} + O(\frac{|\boldsymbol{v}|^2}{c^2})\,, \\ \boldsymbol{B} &= \mu\boldsymbol{H} - \frac{\lambda}{c^2}\boldsymbol{v}\times\boldsymbol{E} + O(\frac{|\boldsymbol{v}|^2}{c^2})\,, \end{aligned} \tag{4}$$

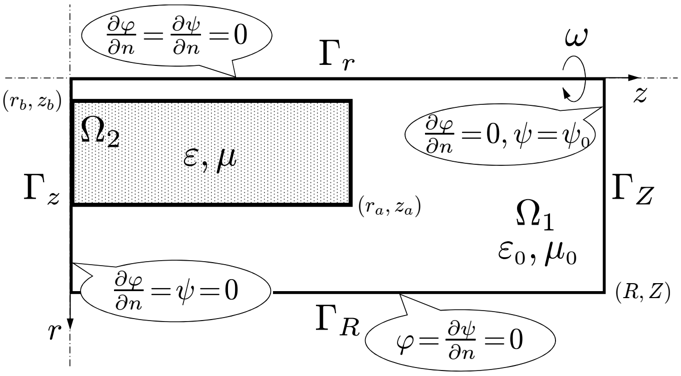

Figure 2. By exploiting the symmetry it is sufficient to consider one quarter Ω_2 of the hollow cylinder's cross section. A rectangular computational domain $\Omega_1 \cup \Omega_2$ has been defined for Finite Element analysis, with the indicated boundary conditions. $(R, Z) = (r_a, z_a) + A\, r_b$; $r_a = 18.65$mm; $z_a = 47.5$mm; $r_b = 10$mm; $z_b = 0$; $A = 2$.

where $c = 1/\sqrt{\varepsilon\mu}$ is the velocity of light in media, and $\lambda = 1 - c^2/c_0^2 \geq 0$ is the *dragging coefficient*. The velocity provides a coupling between the electric and the magnetic fields. Loosely speaking, a moving electric dipole is also perceived as a magnetic dipole, and vice versa.

From Maxwell's equations, the potential ansatz and the constitutive relations we obtain the strong formulation of the problem

$$\mathrm{div} M \mathrm{grad} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} \varepsilon & \lambda\omega r/c^2 \\ -\lambda\omega r/c^2 & \mu \end{pmatrix}. \tag{5}$$

The computational domain and boundary conditions are indicated in Fig. 2. The conditions on Γ_r and Γ_z are dictated to us by symmetry. The solenoid is located outside the computational domain and taken into account by the conditions for ψ on Γ_R ($\partial\psi/\partial n = 0$) and Γ_Z ($\psi = \psi_0$). For simplicity, they are chosen in a way that in the absence of the rotating cylinder there is a homogeneous magnetic field in z-direction. The conditions for φ on Γ_R ($\varphi = 0$) and Γ_Z ($\partial\varphi/\partial n = 0$) correspond to a grounded shield with radius R that extends to infinity in z-direction.

Since the interface and the boundary are non-smooth, we relax the regularity and seek the solution of the problem in $H^1(\Omega_1 \cup \Omega_2)$, which corresponds to the weak formulation: Find $(\varphi, \psi) \in H^1_{0,R} \times (H^1_{\psi_0,Z} \cap H^1_{0,z})$ such that

$$\int_0^{2\pi} \int_{\Omega_1\cup\Omega_2} M \mathrm{grad} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \cdot \mathrm{grad} \begin{pmatrix} \varphi' \\ \psi' \end{pmatrix} \mathrm{d}\Omega\, r \mathrm{d}\varphi = 0 \tag{6}$$

holds $\forall\ (\varphi', \psi') \in H^1_{0,R} \times (H^1_{0,Z} \cap H^1_{0,z})$.[1]

We can show continuity and ellipticity of the bilinear form induced by M in the energy norm

$$\|(\varphi, \psi)\|^2 := \frac{1}{2} \int_0^{2\pi} \int_{\Omega_1\cup\Omega_2} (\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H})\, \mathrm{d}\Omega\, r \mathrm{d}\varphi\,,$$

which yields existence and uniqueness by standard arguments.

[1] $H^1_{x,y} = H^1_{x,y}(\Omega_1 \cup \Omega_2)$ denotes prescribed trace x on Γ_y.

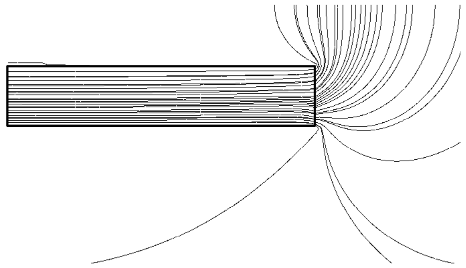

Figure 3. Equipotentials of the electric scalar potential φ. The parameters are chosen as follows: $\mu = 3\mu_0$; $\varepsilon = 6\varepsilon_0$; $\omega = 2\pi \cdot 100\mathrm{s}^{-1}$; mesh size h=1.7mm, ~3.200 degrees of freedom.

III. Simulation

We conducted a Finite Element analysis of the weak formulation, with triangular first order standard nodal elements, compare Fig. 3. The floating potentials due to the metallic coatings on the inner and outer surfaces of the cylinder have been taken into account by a standard Lagrangian multiplier technique. We obtain $V/V_{\mathrm{ref}} \doteq 0.928$, where V is the potential difference, and V_{ref} is a reference potential difference which occurred if the cylinder was conductive.

In the actual experiment, the dragging coefficient λ is determined from the measured potential difference V [1], [2]. Only for an infinite cylinder $z_a \to \infty$ it holds that $V/V_{\mathrm{ref}} \to \lambda \doteq 0.944$ [10].[2] Modeling assumptions have to be taken to deal with the finite axial extension of the cylinder. In the full paper, we will explain how these assumptions can be assessed quantitatively with the help of the Finite Element model.

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