Fast and Robust Method for Mutual Inductance Calculation of Coaxial Circular Coils with Rectangular Cross Section

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***Abstract*—New and extremely fast and precise method for calculating mutual inductance between coaxial circular coils with rectangular cross section in air is presented. It is thoroughly compared with all recently published methods in the same computing environment and the results confirm the above stated. The emphasis in testing has been put on real power transformer conductor dimensions and analysing the accuracy and computational cost the applicability of the presented method is proven. Excellent numerical stability and agreement with professional software was found in all testing cases.**

***Index Terms*—inductance, coils, numerical stability, accuracy, testing**

I. INTRODUCTION

Trying to design numerous electrical devices, one is almost always faced with determining the mutual inductance between circular coils. As a fundamental electrical parameter, it's practical usage appears in numerous fields and applications (all ranges of transformers, generators, motors, current reactors, magnetic resonance applications, antennas, coil guns, medical electronic devices, etc.). Regarding the design principles, circular conductors often have rectangular cross section, primarily in electrical power production and delivery apparatus such as transformers and generators. Since many of the mentioned areas require very precise computational results, especially in medicine, design simplifications such as current loop approximations are not adequate considering it's possibly unreliable outcome. Another key issue, visible mainly in the designing of the devices with numerous conductors such as power transformers, is the computational time needed for obtaining the results. Bearing that in mind, this paper presents the solution for acquiring the mutual inductance between coaxial circular coils with rectangular cross section that is both fast and precise. Throughout recent literature there have been numerous approaches and techniques for solving the same problem, by using elliptic integrals of the first and second kind and Heuman's lambda function [1], using Bessel functions [2] or a combination of Bessel, Struve and Legendre functions [3] or simply using the filament method [1]. Since a completely analytical closed form solution does not exist, each of the methods has to perform numerical integration with respect to at least one variable (cases with two variables also exist [4]). When trying to find the adequate method, a relative amount of effort is needed in order to exploit the solutions in many papers on the subject, since a lot of them have some type of misprints or errors, [4]-[6] to say the least. Four mentioned methods [1]-[4] as well as the filament method have been programmed alongside the proposed one and put in a series of tests with the emphasis on the real power transformer conductor dimensions. All the testing has been done on the same computer using the *Mathematica 9.0* software package. Numerical integration is done using the Gauss-Kronrod quadrature formula wherever applicable. The results are truncated to 6 significant digits and compared both in precision and computing time. The benchmark were results obtained with *Ansoft Maxwell 15.0* professional software package (only 5 significant digits obtainable). The results clearly display the numerical instabilities and extreme amount of computational cost of some methods and the accuracy, stability and low computing time of the presented method.

II. CALCULATION METHOD

The mutual inductance of a pair of turns can be defined by means of the linkage energy. If the current density over its cross sections is assumed uniform, the final result for the inductance is

$$M = \frac{\mu_0}{(R_2-R_1)(Z_2-Z_1)(R_4-R_3)(Z_4-Z_3)} Q, \tag{1}$$

where Q is the quintuple integral

$$Q = \int_{\varphi=0}^{\pi}\int_{z=Z_1}^{Z_2}\int_{Z=Z_3}^{Z_4}\int_{r=R_1}^{R_2}\int_{R=R_3}^{R_4} \frac{rR\cos\varphi dRdrdZdzd\varphi}{\sqrt{r^2+R^2-2Rr\cos\varphi+(z-Z)^2}}. \tag{2}$$

Solving the four integrations with respect to r, R, z and Z analytically with similar technique used in [6], we finally obtain the expression for Q:

$$Q = \int_{\varphi=0}^{\pi} G[R_1,R_2,R_3,R_4,Z_1,Z_2,Z_3,Z_4,\varphi]\cos\varphi d\varphi, \tag{3}$$

where G is

$$\begin{aligned} G = {} & F[R_1,R_3,Z_1,Z_3]-F[R_1,R_3,Z_1,Z_4]-F[R_1,R_3,Z_2,Z_3] \\ & +F[R_1,R_3,Z_2,Z_4]-F[R_1,R_4,Z_1,Z_3]+F[R_1,R_4,Z_1,Z_4] \\ & +F[R_1,R_4,Z_2,Z_3]-F[R_1,R_4,Z_2,Z_4]-F[R_2,R_3,Z_1,Z_3] \\ & +F[R_2,R_3,Z_1,Z_4]+F[R_2,R_3,Z_2,Z_3]-F[R_2,R_3,Z_2,Z_4] \\ & +F[R_2,R_4,Z_1,Z_3]-F[R_2,R_4,Z_1,Z_4]-F[R_2,R_4,Z_2,Z_3] \\ & +F[R_2,R_4,Z_2,Z_4]. \end{aligned} \tag{4}$$

Obviously, function F has an argument φ as a variable, but it is omitted from (4) for the sake of clarity. Finally, function F is

$$
\begin{aligned}
F[r,R,z,Z,\varphi] &= \frac{1}{60}(-10rR(r^3+R^3)\cos^2\varphi+\cos\varphi(5(r^3+ \\
&+R^3)(r^2+R^2+3Y^2)-2r^3RX-2rR^3X)-\frac{X}{2}(-2r^4+ \\
&+r^2(-16R^2+9Y^2)+R^2(-2R^2+9Y^2)+6(r^4+R^4)\cos[2\varphi]+ \\
&+\frac{2Y^4}{\sin^2\varphi})+\frac{1}{2}(\frac{-2Y^5}{\tan\varphi\sin^2\varphi}\arctan[\frac{Y^2\cot\varphi+rR\sin\varphi}{YX}]- \\
&-30r^2R^2Y\log[Y+X]+15(r^4+R^4)Y\cos[2\varphi]\log[Y+X]+ \\
&+\cos\varphi(5r^3(r^2-4Y^2)\log[R-r\cos\varphi+X]+5R^3(R^2-4Y^2)\cdot \\
&\cdot\log[r-R\cos\varphi+X]+2r^5\log[\frac{\sqrt{Y^2+r^2\sin^2\varphi}}{R-r\cos\varphi+X}]+2R^5\cdot \\
&\cdot\log[\frac{\sqrt{Y^2+R^2\sin^2\varphi}}{r-R\cos\varphi+X}])-\cos[3\varphi](r^5(5\log[R-r\cos\varphi+X]+ \\
&+2\log[\frac{\sqrt{Y^2+r^2\sin^2\varphi}}{R-r\cos\varphi+X}])+R^5(5\log[r-R\cos\varphi+X]+ \\
&+2\log[\frac{\sqrt{Y^2+R^2\sin^2\varphi}}{r-R\cos\varphi+X}]))-5Y(r^4(4\arctan[\frac{Y}{r\sin\varphi}]+ \\
&+3\arctan[\frac{Y(r\cos\varphi-R)}{r\sin\varphi X}])+R^4(4\arctan[\frac{Y}{R\sin\varphi}]+ \\
&+3\arctan[\frac{Y(R\cos\varphi-r)}{R\sin\varphi X}]))\sin[2\varphi])), \qquad (5)
\end{aligned}
$$

where $Y=z-Z$ and $X=\sqrt{r^2+R^2+Y^2-2rR\cos\varphi}$.

III. Testing

The testing was done on three different geometric arrangements (0-1, 0-2, 0-3), as can be seen in Fig. 1. Positions correspond to the usual arrangements of conductors inside the power transformer coil. The width and height of the conductor were chosen based on the real industrial data. More thorough testing will be presented in the full paper. All examples were run for three consecutive times for each of the methods and the mean time required for evaluation is given.

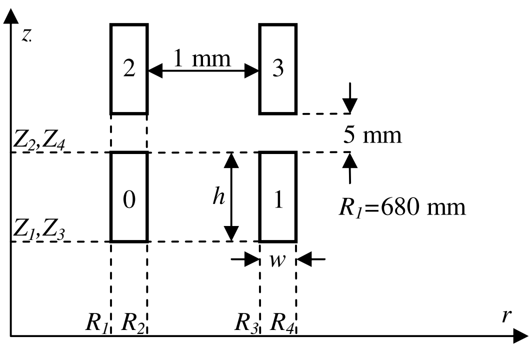

Fig. 1. Magnetization as a function of applied field

TABLE I
COMPARISON OF OBTAINED RESULTS

Method	**mutual ind (μH)**	**time (s)**
w = 1 mm, h = 2 mm	**0-1 arrangement**	
Ansoft Maxwell 15.0	5.0139	/
Presented method	5.01385	0.11
Conway [3]	5.42041	47.91
Babić et al.[1]	4.92752	0.28
Ravaud et al. [4]	5.01387	0.82
Hurley et al. [2]	5.16349	1.53
Filament 1/1/1/1	5.01833	0.00
Filament 5/5/5/5	5.01420	0.30
Filament 10/10/10/10	5.01396	3.93
Filament 20/20/20/20	5.01389	56.44
w = 1 mm, h = 10 mm	**0-2 arrangement**	
Ansoft Maxwell 15.0	3.3652	/
Presented method	3.36521	0.06
Conway [3]	3.36520	220.09
Babić et al.[1]	4.89574	0.25
Ravaud et al. [4]	3.36521	0.84
Hurley et al. [2]	3.36531	1.26
Filament 1/1/1/1	3.36074	0.00
Filament 5/5/5/5	3.36486	0.30
Filament 10/10/10/10	3.36511	3.91
Filament 20/20/20/20	3.36518	56.65
w = 8 mm, h = 25 mm	**0-3 arrangement**	
Ansoft Maxwell 15.0	2.7800	/
Presented method	2.77996	0.10
Conway [3]	2.77995	211.68
Babić et al.[1]	2.78003	0.26
Ravaud et al. [4]	2.77996	0.74
Hurley et al. [2]	2.77990	2.84
Filament 1/1/1/1	2.77581	0.00
Filament 5/5/5/5	2.77965	0.30
Filament 10/10/10/10	2.77988	3.92
Filament 20/20/20/20	2.77994	56.73

IV. Conclusion

As can be seen in Table I, the results confirm that the presented method is both very accurate and extremely fast. Some other methods reveal instabilities and lack of accuracy. With speed improvements ranging from at least 3 to a couple of thousand times, other noteworthy benefit of this method is the easiness of implementation, since all mathematical functions used are trivially applied in any programming language.

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