Linear Subspace Reduction for Quasistatic Field Simulations

Daniel Schmidthäusler[1], Sebastian Schöps[2,3], Markus Clemens[1], *Senior Member, IEEE*

[1] Bergische Universität Wuppertal, FB E, Chair of Electromagnetic Theory,
Rainer-Gruenter-Str. 21, D-42119 Wuppertal, Germany
[2] Technische Universitaet Darmstadt, Institut fuer Theorie Elektromagnetischer Felder,
Schlossgartenstr. 8, D-64289 Darmstadt, Germany
[3] Technische Universitaet Darmstadt, Graduate School of Computational Engineering,
Dolivostr. 15, D-64293 Darmstadt, Germany

Abstract—**Simulating eddy-current problems in time domain using the magnetic vector potential formulation leads to a non-linear system of differential-algebraic equations. Typically a large number of degrees of freedom (DoF) are in domains with constant material, for example air or vacuum in exterior domains. In this work a new method is presented that reduces only the DoFs in these exterior domains by using the proper orthogonal decomposition (POD) method. The POD method involves a singular value decomposition (SVD) to capture the system dynamics and extract the essential dynamical behavior with a low number of DoFs. A simple transformer example is given to proof the concept.**

Index Terms—**Model order reduction, eddy current problem, finite element method.**

I. INTRODUCTION

Low frequency problems that exhibit inductive and resistive effects are typically modeled by the magneto-quasistatic (MQS) approximation. The resulting *eddy current problem* is a non-linear parabolic partial differential equation with boundary and initial conditions. Although using finite elements (FE) with adaptive and unstructured meshes, the semi-discrete problems still consist of a large number of degrees of freedom (DoFs). In particular, if very similar problems are solved repeatedly, e.g., in an optimization loop or to quantify uncertainties, a reduction of the DoFs is necessary. One way out is FE coupled to Boundary Elements [1]. Furthermore model order reduction (MOR), e.g., the proper orthogonal decomposition (POD). Most MOR techniques involve a singular value decomposition (SVD) to capture the system dynamics and extract the essential dynamical behavior with few DoFs. This was successfully applied to (linear) problems in the time domain, e.g., electro-[1] and recently (non-linear) magneto-quasistatic [3].

In this paper we propose a new approach that reduces only the linear system parts of the quasistatic system, e.g. the eddy current problem. This avoids the main difficulty of classical MOR techniques with non-linear problems: the assembly of the full system of equations in every Newton step. A similar approach for electro-quasistatic problems was shown to reduce the DoFs considerably [4].

II. MAGNETO-QUASISTATIC FINITE ELEMENT FORMULATION

The application of Whitney finite elements [5] to the magneto-quasistatic initial-boundary value problem in magnetic vector potential formulation yields the semi-discrete problem
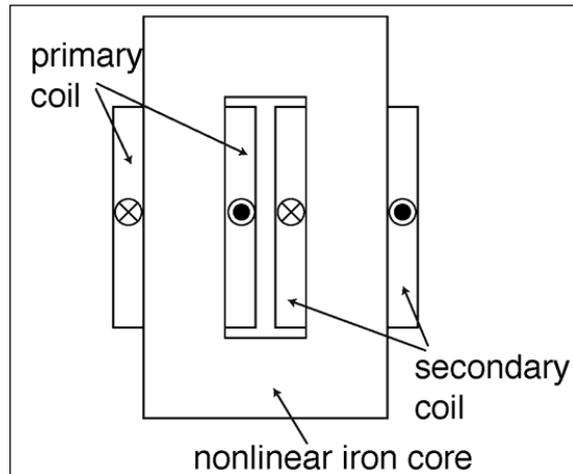
Fig. 1 2D transformer geometry with conductive non-linear iron core.

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a} + \mathbf{K}(\mathbf{a})\mathbf{a} = \mathbf{j}_s \qquad (1)$$

where $\mathbf{a}$ is the (line integrated) discrete magnetic vector potential, $\mathbf{M}$ the singular conductivity matrix and $\mathbf{K}(\mathbf{a})$ is the non-linear curl-curl reluctance matrix that accounts for magnetic saturation effects.

For simplicity we assume that only electrically conductive parts ($\sigma > 0$) exhibit a non-linear magnetic saturation behavior. Sub-structuring the spatial domain according to the conductive and non-conductive subdomains corresponds to a coupled formulation that consists of a quasistatic and static problem [5]

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\mathbf{a}) & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{j}_1 \\ \mathbf{j}_2 \end{bmatrix} \qquad (2)$$

where the non-conductive part has a constant reluctance, i.e., $\mathbf{K}_{22}$ is constant. Finally, time integration, e.g., by using the implicit Euler scheme, leads to a series of non-linear systems for each time step t_i. They are commonly solved by Newton's method which yields discrete solutions $\mathbf{a}^i$.

III. MODEL ORDER REDUCTION

The POD is a method for building low-dimensional approximations of both linear and non-linear systems, [7]. It
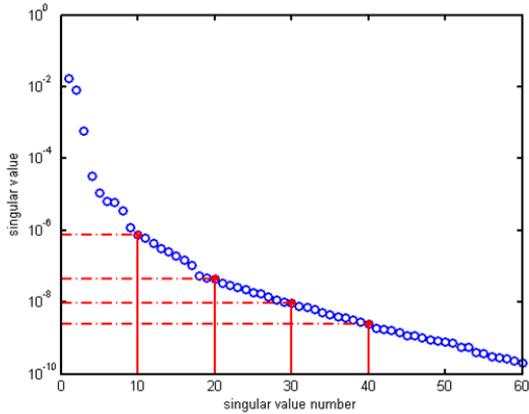
Fig. 2 Singular values with marking 10th, 20th, 30th and 40th.

Fig. 3 Max. relative error of flux magnitude in time steps.

extracts the dynamical behavior from observations of the system variables. The observations are assembled in a *snapshot* matrix

$$\mathbf{X} = \left[\mathbf{a}^1,\dots,\mathbf{a}^N \right] . \tag{3}$$

that consists of previous solutions that may stem from another but similar problem, e.g., from a linearized or frequency domain formulation.

Applying the SVD to $\mathbf{X}$ results in the rank-one decomposition of the snapshot-matrix

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}} = \sigma_1\mathbf{u}_1\mathbf{v}_1^{\mathrm{T}} + \dots + \sigma_p\mathbf{u}_p\mathbf{v}_p^{\mathrm{T}} \tag{4}$$

where $\mathbf{U}$ and $\mathbf{V}^{\mathrm{T}}$ are orthonormal matrices and the singular values σ_i are sorted in descending order. The columns of $\mathbf{U}$ contain the directions where the system dynamics occur and the singular values weight these directions. Approximating the dynamics means to choose the first r most weighted columns of $\mathbf{U}$. This results in a reduced order basis

$$\mathbf{U}_r = \left[\mathbf{u}^1,\dots,\mathbf{u}^r \right] . \tag{5}$$

Applying this low order approximation only for the non-conductive linear part of the system (2) yields the reduced order system

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\mathbf{a}) & \mathbf{K}_{12}\mathbf{U}_r \\ \mathbf{U}_r^T\mathbf{K}_{21} & \mathbf{U}_r^T\mathbf{K}_{22}\mathbf{U}_r \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{j}_1 \\ \mathbf{U}_r^T\mathbf{j}_2 \end{bmatrix} . \tag{6}$$

Thus the magnetostatic part of the coupled problem is calculated in reduced form. The reduced matrix $\mathbf{U}_r^T\mathbf{K}_{22}\mathbf{U}_r$ and the off-diagonal coupling matrices have to be assembled once before starting the time transient simulation. During this process only the contributions of the elements with non-linear material laws have to be assembled.

IV. COMPUTATIONAL EXAMPLE

The example showing the method in action is a standard 2D transformer model shown in Fig. **1** which is 30 x 40 cm in size with 358 primary and 206 secondary copper strands [5]. The model is discretized with FEMM [8] using 7713 nodes of which 3363 are in the linear subdomain.

To provide a proof of concept the assembly of the snapshot matrix was done by a full simulation. In the full paper the possibility of more efficient snapshot assembly implementation, e.g., time harmonic solutions and the usage of the full solution assembled snapshot matrix applied to material varied simulations will be discussed. The 60 first singular values from a full simulation are shown in Fig. **2**. The linear subdomain projection was applied for 40, 30, 20 and 10 singular values, which determines the reduced dimension of the subdomain. In Fig. **3** the maximal relative error of the flux magnitude at every time step shows that a reduction factor of more than 100 for the subdomain can be reached. A speedup and scaling exploration for models with more DoF will be discussed in the full paper.

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