Second Moment Analysis of the Nonlinear Eddy Current Model with Material Uncertainties

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***Abstract*—In this paper we study the eddy current problem with uncertainties in the nonlinear material characteristic originating, e.g., from measurements. In the case of small input uncertainties adjoint techniques can be used to efficiently approximate the solutions statistics (first order second moment method) at very low computational cost. We will carry out the corresponding sensitivity analysis and investigate the methods approximation properties. The main contribution of the present work is the extension to nonlinear problems. Numerical results for an electrical transformer and a comparison with the standard Monte Carlo method are given.**

***Index Terms*—uncertainty quantification, eddy current model, sensitivity analysis**

I. Introduction

To obtain reliable simulation results it is important to quantify uncertainties. In particular the nonlinear material relation of ferromagnetic materials in the eddy current model is determined through measurements that contain errors. Furthermore, in practice the material relation is perturbed by the manufacturing process, [1], and the influence on the quantities of interest (QOI) should be investigated. In many situations the statistical moments are sufficient to characterize the systems (uncertain) behavior. To compute these moments Monte Carlo methods have been frequently used in the past. Due to its non-intrusive character Monte Carlo simulation does not require any modification of the simulation code. However, the main drawback is the high computational cost, i.e., typically the underlying problem has to be solved several thousand times. More efficient alternatives are, e.g., based on generalized polynomial chaos, but still the computational cost can be high, in particular for a large number of uncertain inputs.

When the input uncertainties are sufficiently small the moments of the QOIs, e.g., the mean and the variance, can be efficiently approximated through *second moment analysis*, [2]. Requiring only the solution of one deterministic and its adjoint problem the computational cost of the aforementioned methods can be reduced significantly at the drawback of being an intrusive method. Additionally, the method easily handles a large number of uncertain inputs. For the analysis of moment based methods for linear problems we refer to, e.g., [3], [4]. The main contribution of the present work is their extension to nonlinear problems.

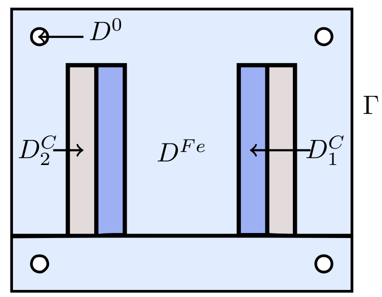

Fig. 1 2D model geometry. The domain D is composed of vacuum D^0, a ferrromagnetic material D^{Fe} and the primary/secondary coil windings $D^C_{1;2}$. FEMM example taken from [6].

II. Model Problem

We consider the transient magnetoquasistatic problem

$$\begin{cases} \textbf{curl}\,\mathbf{E} = -\partial_t \mathbf{B}, \\ \textbf{curl}\,\mathbf{H} = \sigma\mathbf{E} + \mathbf{J}. \end{cases}$$

In a 2D setting, referring to Fig. 1: let a domain D with boundary Γ be decomposed into parts filled with a ferromagnetic material D^{Fe}, vacuum D^0 and coil parts $D^C_{1;2}$, respectively. Denoting $u = A_z$ for the longitudinal component of the magnetic vector potential and introducing $\boldsymbol{\eta}(\mathbf{u}) := \nu(|\mathbf{u}|)\mathbf{u}$ for the sake of readability, the weak formulation is to find $u \in V$, where V is an appropriate space, subject to

$$\int_D \sigma\partial_t uv \ dx + \int_D \boldsymbol{\eta}(\textbf{grad}\, u)\cdot\textbf{grad}\, v \ dx = \int_D Jv \ dx, \ \forall v \in V,$$

endowed with the initial condition $u(0,x) = u_0(x)$, in D and homogeneous Dirichlet boundary conditions on Γ. Uncertainties are introduced through the probability space (Ω, Σ, P). The set of random realizations Ω is such that $\nu(\omega,\cdot)$, $\omega \in \Omega$, gives a physical admissible material curve. We assume for the stochastic solution $u \in L^2(\Omega, V)$ to guarantee finite second moments. Focussing on the nonlinear material relation we do not model an uncertain conductivity.

III. Deterministic Second Moment Analysis

A. Approximation of Mean and Variance

Many QOIs, e.g., energies or the mean inductance, can be written in a general form as

$$F = \int_{I_T}\int_D f(t,u,\partial_t u) \ dxdt,$$

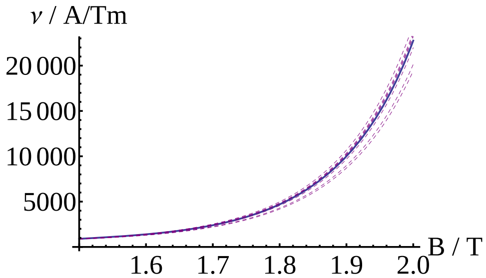

Fig. 2 Random Brauer model. Mean (solid) and random realizations (dashed).

with I_T being the time interval of interest. Given the uncertainty in the form $\nu_\delta(\omega) = \bar{\nu} + \delta\tilde{\nu}(\omega)$, with mean $\bar{\nu}$, we expand using Taylor's series

$$F(\nu_\delta(\omega)) = F(\bar{\nu}) + \delta dF(\bar{\nu};\tilde{\nu}(\omega)) + \mathcal{O}(\delta^2).$$

Under the assumption $E[\tilde{\nu}] = 0$, it can be shown [2] and will be discussed in more detail in the full paper that

$$E[F] = F(\bar{\nu}) + \mathcal{O}(\delta^2),$$
$$Var[F] = \delta^2 E[dF(\bar{\nu};\tilde{\nu}(\omega))^2] + \mathcal{O}(\delta^3).$$

Thus, computing $E[dF(\bar{\nu};\tilde{\nu}(\omega))^2]$ we get a second order approximation of the mean and a third order approximation of the variance of the QOI in terms of the perturbation amplitude δ, respectively.

B. Sensitivity Analysis

For brevity, we only state that the gradient is given by

$$dF(\nu;\tilde{\nu}) = -\int_{I_T}\int_{D^{Fe}} \tilde{\boldsymbol{\eta}}(\mathbf{grad}\, u)\cdot\mathbf{grad}\,\xi\ dxdt,$$

where $\tilde{\boldsymbol{\eta}}(\mathbf{u}) = \tilde{\nu}(|\mathbf{u}|)\mathbf{u}$. This in turn requires the solution ξ of the adjoint problem

$$\begin{cases} -\sigma\partial_t\xi - \operatorname{div}\boldsymbol{\eta}_L(\mathbf{grad}\,\xi) = f_y(\cdot,u,\partial_t u) - \partial_t f_z(\cdot,u,\partial_t u), \\ \xi(T) = \xi_{end}, \end{cases}$$

with linearization $\boldsymbol{\eta}_L(\mathbf{grad}\,\xi)$, where f_y and f_z denote the derivatives with respect to the second and third variable, respectively. Finally, we can show with

$$h(t,x) := \mathbf{grad}\, u(t,x)\cdot\mathbf{grad}\,\xi(t,x)$$
$$c(t,x,s,y) := Cor_{\tilde{\nu}}(|\mathbf{B}(t,x)|,|\mathbf{B}(s,y)|)h(t,x)h(s,y),$$

and Cor being the two-point correlation function, that

$$E[dF(\bar{\nu};\tilde{\nu}(\omega))^2] = \iint_{I_T\times D^{Fe}}\iint_{I_T\times D^{Fe}} c(t,x,s,y)\ dxdt\ dyds. \quad (1)$$

IV. Numerical Results

We restrict ourselves to the case $\sigma = 0$. Let the reluctivity be given by Brauer's model [5]

$$\nu(B) = k_1 e^{k_2 B^2} + k_3.$$

Through linearization of the Brauer model we get

$$\nu_\delta(\omega,B) = \bar{k}_1 e^{\bar{k}_2 B^2} + \bar{k}_3 + 2\delta(dk_1(\omega)e^{B^2\bar{k}_2} + dk_2(\omega)B^2 e^{B^2\bar{k}_2}\bar{k}_1 + dk_3(\omega)).$$

TABLE 1
Perturbation analysis vs. Monte Carlo (3375 samples).

	δ	E[L]	Var[L]
Monte Carlo	0.1	6.58444	0.02952
	0.05	6.56306	0.00843
	0.025	6.53262	0.00195
Perturbation analysis	0.1	6.53942	0.03009
	0.05	6.53942	0.00752
	0.025	6.53942	0.00188

We model dk_1, dk_2, dk_3 to be independent random variables with uniform distribution, i.e., $dk_i \propto \mathcal{U}(-0.5, 0.5)$. Note that in a more realistic setting these parameters should be considered to be correlated. Fig. 2 shows the mean value and some random realizations for $\mathbf{k} = (3.8, 2.17, 396.2)$. However, in this setting the two-point correlation function is

$$Cor_{\tilde{\nu}}(x,y) = \frac{1}{12}(1 + e^{\bar{k}_2(x^2+y^2)}(1 + \bar{k}_1^2 x^2 y^2)).$$

We use first order nodal finite elements and an in-house MATLAB code for the simulation. The domain is triangulated with 4571 nodes using FEMM [6]. Here, the QOI is the chord inductance L of the primary coil. The numerical results are given in Tab. 1. We observe that the second moment perturbation analysis agrees with the reference solution (Monte Carlo) up to 1%, 3% and 10% in the variance for the amplitudes δ. While the Monte Carlo simulation requires a few thousand simulations, the perturbation analysis requires only two solutions of the deterministic and adjoint problems and the higher dimensional quadrature (1). However, the computational costs of this quadrature are negligible for linear elements.

V. Conclusion and Outlook

In this paper we proposed to use a second moment perturbation analysis as a cheap alternative to Monte Carlo simulation. The results of the given example indicate that we can approximate the variance accurately at a drastically reduced cost. In the full paper the time-transient case will also be illustrated by an example.

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