On Forces in Magnetized Matter

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***Abstract*—In magneto-elastic interactions, the Virtual Power Principle is powerful enough to solve all force problems, but only if the *coupled* constitutive laws are known integrally. We discuss the problem of forces in magnetized matter, insisting on the fact that a full knowledge of the magnetic field may *not* be enough to determine the force density when the local B–H law depends on the local strain of the material.**

***Index Terms*—Electromagnetic forces, Maxwell tensor, magnetostriction.**

I. Introduction

Forces in magnetized matter are not yet properly understood. Not, at least, from the point of view of engineers involved in calculations. We should like to have formulas that, once the field has been computed by finite elements or other numerical techniques, could be coded in order to compute the force density from the fields data. There is no dearth of such formulas, but their amazing diversity is disturbing: Are all these proposals equivalent, in spite of formal differences, or are they substantially distinct, being based on different theories?

Part of the difficulty—the mildest part—is notational: It could be the case that two formulas are secretly the same, being reducible one to the other by a few lines of vector algebra, but checking that may be difficult when authors don't explicitly state what they mean by such formulas as $B \cdot \nabla M$ or $\nabla M \cdot B$. We shall try here to lift such ambiguities.

More seriously, theories on force density are often based on microscopic models: For instance, magnetized matter is construed as a distribution of magnetic dipoles, and since the mechanical effect of the field on such dipoles is known, one may think that summing up these elementary forces solves the problem. But what if different microscopic models then appear to *disagree*? Which does happen, as we well know: The field M inside a hard magnet can be ascribed to magnetic charges or to Amperian currents, with identical results as regards the total force and torque on the magnet, but with spectacularly different predictions about the force field, and hence, about the eventual deformation of the magnet. So one should not rely on imagined microscopic mechanisms, to develop the theory, only on measurable macroscopic properties: The virtual power principle (VPP) makes this possible, and points to the determination of the energy density as a function of *both* the electromagnetic field (EM field) *and* the material deformation as the central problem in the question of forces.

At this stage, one realizes that knowing the EM field may not be enough to compute the force field: a complete description of the *coupled* constitutive laws may be required. As a corollary, the Maxwell tensor does not know all about forces, which challenges its (alleged by some) status as cornerstone of the theory: Indeed, there are cases when the force density differs from the divergence of the Maxwell tensor. This is the hallmark of *magnetostriction*: The state of affairs when local magnetic properties such as permeability or magnetization depend on the *local* deformation of the body.

The purport of the present paper is to unfold this theory from first principles, using the lightest possible mathematical apparatus. In particular, the "material form" [1] of the equations and the differential-geometric formalism that naturally goes with it [2] are avoided, in favor of a Eulerian treatment with familiar vector entities only. We first reestablish a standard result: Force is equal to $J \times B$ minus the derivative of magnetic energy, expressed as a function of B and the displacement field, with respect to the latter. Then we treat a few simple cases, enough to show how easily can magnetostriction be overlooked, and how the formalism allows to take it into account when needed.

II. General expression of the force field

We work in 3D Euclidean space, with dot product $X \cdot Y$. A material particle lying at point x at time $t = 0$ will be found at point $x + u(t, x)$ at time t, the displacement u being a smooth vector-valued function of x and t, null for $t = 0$. (We don't want initially distinct particles to collide, so the correspondence $x \to x + u(t, x)$ should be 1–1: This is so for $t > 0$ small enough.) The velocity field v (set in roman to avoid possible confusion with reluctivity ν) is the time derivative $\partial_t u$. Only its value at time 0 will matter when invoking the VPP. One assumes $u(t, x)$ uniformly bounded and null outside some bounded region of space that contains all the matter and currents participating in the interaction. (For convenience, points x of this region that lie in the air are also assigned a displacement, whose value will play no role, provided u stays smooth.) A source-current density $J^s(t, x)$, maintained by some exterior agency, is given, also null far away. Time-dependent fields E, H, B describe the electromagnetic situation. (The displacement current D is ignored, as well as electric charge, which entails the neglect of Coulomb forces.)

We adopt the time primitive (up to sign) of the electric field, $A(t) = A_0 - \int_0^t E(s)\, \mathrm{d}s$, as field descriptor. Thus, $E = -\partial_t A$ and $B = \mathrm{rot}\, A$. Conductivity and reluctivity (more convenient here than its inverse, the permeability μ) will depend on the displacement u, so we denote them by σ_u and ν_u, without being more specific for the time being. A typical form of the evolution equation is, with initial condition $A(0) = A_0$,

$$\sigma_u(\partial_t A - \mathrm{v} \times B) + \mathrm{rot}(\nu_u \,\mathrm{rot}\, A) = J^s, \tag{1}$$

but to gain some generality we shall introduce the magnetic energy $\Psi(u, A)$, equal to ${}^1\!/_2 \int \nu_u\, |\mathrm{rot}\, A|^2$ in the case of (1). (The integration, here and in what follows, is over all space.) This way the partial Fréchet derivative $\partial_A\Psi$ is the vector field $\mathrm{rot}\, H$, where $H = \nu_u \,\mathrm{rot}\, A = \nu_u\, B$, so the equation becomes

$$\sigma_u(\partial_t A - \mathrm{v} \times B) + \partial_A\Psi(u, A) = J^s, \tag{2}$$

which covers the nonlinear case of ferromagnetic (but non-hysteretic) materials.

Since Ψ appears here as just a device to formulate the magnetic law, we should justify calling it magnetic *energy*. For this, remark that the rate of change of $\Psi(u, A)$ is, by the chain rule,

$$\mathrm{d}_t[\Psi(u(t), A(t))] = \int \partial_u\Psi \cdot \mathrm{v} + \int \partial_A\Psi \cdot \partial_t A. \tag{3}$$

Next, taking $\mathrm{v} = 0$ in (2), dot-multiply both sides of it by $\partial_t A$ and integrate over space. This results in

$$\int \sigma_u|\partial_t A|^2 + \mathrm{d}_t[\Psi(u, A(t))] = -\int J^s(t) \cdot E(t). \tag{4}$$

The right-hand side of (4) is the power brought into the system by the source current, and $\int \sigma_u\, |\partial_t A|^2 = \int \sigma_u\, |E|^2$ is the Joule loss. So the second term on the left represents the fraction of power that must be stored in the magnetic field, which is the needed justification: With the convention that $\Psi(u, 0) = 0$ whatever u, the quantity $\Psi(u, A)$ appears as the magnetic energy of the field $B = \mathrm{rot}\, A$ in configuration u.

Now, return to the case where u can evolve in time. Denoting by $J = \sigma_u(-\partial_t A + \mathrm{v} \times B)$ the induced current in conductors, Joule losses are

$$\int \sigma_u|\partial_t A - \mathrm{v} \times B|^2 = \int \sigma_u(\partial_t A - \mathrm{v} \times B) \cdot \partial_t A - \int \mathrm{v} \cdot (J \times B). \tag{5}$$

Repeat the above process—dot-multiply both sides of (2) by $\partial_t A$ and integrate in space. Combining (3) and (5), one gets

$$\begin{aligned} &\int \sigma_u|\partial_t A - \mathrm{v} \times B|^2 + \mathrm{d}_t[\Psi(u, A)] + \\ &\cdots + \int \mathrm{v} \cdot (J \times B - \partial_u\Psi(u, A)) = -\int J^s \cdot E. \end{aligned} \tag{6}$$

Considering v here as the velocity in a virtual motion, the third term on the left appears as the corresponding virtual power. The force field, therefore, is $J \times B - \partial_u\Psi(u, A)$.

The full paper will show how the vector field $\partial_u\Psi$ is computed in practice, in a series of important cases: Non-homogeneous B–H law, linear or non-linear, isotropic or anisotropic, permanent magnets. This summary must restrict to the first topic. Yet there is room for a remark of general validity: We are interested in the force field in the reference configuration, the one in which $u = 0$, so we only want $\partial_u\Psi(u, A)$ at $u = 0$ (and $A = A_0$, the initial condition). In such a case the following trick, based on the concept of "directional" derivative, is available: For each vector field v (conceived here as a virtual velocity field), set $u = t\mathrm{v}$, and find the limit of $[\Psi(t\mathrm{v}, A_0) - \Psi(0, A_0)]/t$ when t tends to 0. This limit, which is also the derivative in t of $\Psi(t\mathrm{v}, A_0)$ at $t = 0$, has the form $\int \mathrm{v} \cdot \partial_u\Psi(0, A_0)$, from which what we shall call the "extra force field", $f = -\partial_u\Psi(0, A_0)$, can be read off.

III. Computing $\partial_u\Psi$: An example

In the simplest case, $\nu = 1/\mu_0$ all over, one has $\Psi(u, A) = {}^1\!/_2 \int \mu_0^{-1}\, |\mathrm{rot}\, A|^2$, which does not depend on u, so $f = 0$, and the only force is $J \times B$, as expected.

Next, suppose that, at time 0, point x is occupied by some material the reluctivity of which we shall denote by $\nu_{\mathrm{mat}}(x)$. Given the virtual velocity field $x \to \mathrm{v}(x)$, let us build the virtual displacement $u(t, x) = t\mathrm{v}(x)$. Call ν_t the evolving reluctivity at time t, equal to ν_{mat} at $t = 0$. Since the particle that occupied point x at time 0 has reached the point $x + t\mathrm{v}(x)$ at time t, it seems natural (but be wary there!!) to assert that $\nu_t(x + t\,\mathrm{v}(x)) = \nu_{\mathrm{mat}}(x)$. Differentiating in t this equality, we find that $\partial_t\nu_t + \nabla\nu_t \cdot \mathrm{v} = 0$ at all points. Magnetic energy at t being $\Psi(t\mathrm{v}, A_0) = \frac{1}{2}\int \nu_t(x)\, |(\mathrm{rot}\, A_0)(x)|^2\, \mathrm{d}x$, where $\mathrm{d}x$ denotes the volume element, its time-derivative at $t = 0$ is $-\frac{1}{2}\int (\mathrm{v}(x) \cdot \nabla\nu_{\mathrm{mat}}(x))\, |(\mathrm{rot}\, A_0)(x)|^2\, \mathrm{d}x$. The extra force field, $-\partial_u\Psi(u, A_0)$, is therefore $f(x) = {}^1\!/_2\, \nabla\nu_{\mathrm{mat}}(x)\, |(\mathrm{rot}\, A_0)(x)|^2$, or better, getting rid of the notational clutter, $f = {}^1\!/_2\, \nabla\nu\, |\mathrm{rot}\, A|^2$, or else, ${}^1\!/_2\, |B|^2\, \nabla\nu$. Since $B = \mu H$ and $\nu\mu = 1$, this is the same as $-{}^1\!/_2\, |H|^2\, \nabla\mu$ (known to Helmholtz, according to Carter [3]), more often encountered in the literature (cf., e.g., [3][4][5]).

But natural as this may look, assuming that the reluctivity of a matter's chunk does not change as it moves is an assertion about the constitutive law that should be acknowledged and experimentally tested. It may happen that ν changes with compression or extension—an example of magnetostrictive behavior. If so, one must set $\nu_t(x + t\,\mathrm{v}(x)) = \nu_{\mathrm{mat}}(x, t\,\mathrm{div}\,\mathrm{v}(x))$ with a different, more informative ν_{mat}, now a function $\nu_{\mathrm{mat}}(x, \rho)$ of *two* variables, ρ being a placeholder for the volumic expansion, which is $\mathrm{div}\, u$ for a displacement u. (The ratio of volumes is $1 + \mathrm{div}\, u$.) This time, $\partial_t\nu_t + \nabla\nu_t \cdot \mathrm{v} = \partial_\rho\nu_{\mathrm{mat}}\,\mathrm{div}\,\mathrm{v}$, so the t-derivative of $\Psi(t\mathrm{v}, A_0)$ at $t = 0$ is one-half of

$$-\int [(\mathrm{v}(x) \cdot \nabla\nu_{\mathrm{mat}}(x, 0) - \partial_\rho\nu_{\mathrm{mat}}(x, 0)\,\mathrm{div}\,\mathrm{v}(x)]\, |(\mathrm{rot}\, A_0)(x)|^2\, \mathrm{d}x.$$

An integration by parts of the term containing div v leads to the final form of the extra force field, ${}^1\!/_2\, |B|^2\, \nabla\nu + \nabla(\partial_\rho\nu\, |B|^2/2)$. The last part of it is easily overlooked.

No armchair physics can give us the value of $\partial_\rho\nu$. It must be measured! Note also that $\nu(x) = \nu_{t=0}(x) = \nu_{\mathrm{mat}}(x, 0)$, so the fields B and H are the same whatever the dependence of ν_{mat} on ρ. So the force density is not determined by them. This implies that the divergence of the Maxwell tensor, which only depends on B and H, would miss the $\nabla(\partial_\rho\nu\, |B|^2/2)$ part of the force field, which supports our introductory claim. We'll find other supporting examples in the full-length paper.

References

[1] M. Lax and D. F. Nelson, "Maxwell equations in material form", Phys. Rev. B, vol. 13, no. 4, pp. 1777-1784, 1976.

[2] A. Bossavit, "Differential forms and the computation of fields and forces in Electromagnetism", Eur. J. Mech., B/Fluids, vol. 10, no. 5, pp. 474-488, 1991.

[3] G.W. Carter, "Distribution of mechanical forces in magnetised material", Proc. IEE, vol. 112, no. 9, pp. 1771-1777, 1965.

[4] F.N.H. Robinson, Macroscopic Electromagnetism, Oxford: Pergamon Press, 1973.

[5] G. Maugin, "The principle of virtual power in continuum mechanics: Application to coupled fields", Acta. Mech., vol. 35, no. 1/2, pp. 1-70, 1980.