Convolution-Free Modelling of Dispersive Media in the Time-Domain Finite-Element Solution of the Vector Wave Equation

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***Abstract*—A new approach to derive finite-element time-domain (FETD) formulations for modelling a general linear dispersive media is developed. The formulation starts from the mixed FETD discretized by Whitney forms in space and the Crank-Nicolson (CN) method in time. The effect of the media dispersion is taken into account separately using a bilinear transformation approach. Having combined these equations, a formulation is reached that reduces to the second-order vector wave equation discretized by the Newmark-β method with $\beta = 1/4$ in non-dispersive media. So, our formulation can be considered as an extension of the unconditionally stable (US) vector wave equation to dispersive media using a bilinear transformation method. Therefore, it does not suffer from intrinsic limitations of the existing convolution-based formulations. Furthermore, it can be concluded that the CN-FETD is a mixed version of the vector wave equation discretized by Whitney forms in space and the US Newmark-β method in time. Finally, the formulation is applied to obtain reflection and transmission coefficients of a doubly dispersive dielectric slab and highly accurate results are reached.**

***Index Terms*—Dispersive media, Finite element methods, Time domain analysis.**

I. Introduction

Modelling dispersive media has attracted considerable attention in recent years due to the versatility of the practical applications involving materials such as metamaterials, snow, soils and biological tissues [1]. Significant efforts have been made to extend the finite-difference time-domain (FDTD) method to dispersive media [2]-[4]; however, only a few papers have been dedicated to the FETD [5]-[7]. To the best knowledge of the authors, all of the extensions of the US vector wave equation to dispersive media are based on the convolution-based methods, which are not capable of modelling either arbitrary dispersive or non-linear media. In addition, computational cost becomes cumbersome for high-order media. Although the first problem has been alleviated using recursive fast-Fourier transforms [7], it still suffers from the other limitations. Recently, Donderici and Teixeira have developed a formulation based on the mixed FETD using the auxiliary differential equation (ADE) approach [8], but it is only conditionally stable.

In this paper, we introduce a new formulation that can be considered as an extension of the US vector wave equation to dispersive media. Although no proof is provided to demonstrate unconditional stability of the method, numerical simulation verifies this property. In addition, it exactly reduces to the vector wave equation discretized by the provably US Newmark-β method with $\beta = 1/4$ in non-dispersive media [9]. Since a bilinear transformation method has been used to derive this formulation [4], it does not have the limitations of the previously developed methods for the vector wave equation.

II. Formulation

Consider the Maxwell equations in which the electric field and the magnetic field are expanded with the 1-form and 2-form Whitney elements in space

$$\frac{\partial\{d\}}{\partial t} = [C]^T\{h\}, \frac{\partial\{b\}}{\partial t} = -[C]\{e\} \tag{1a}$$

$$\{d\} = \varepsilon[\mathcal{M}]\{e\}, \{h\} = \mu^{-1}[\mathcal{M}_f]\{b\} \tag{1b}$$

where $\{.\}$ denotes an unknown vector. $[\mathcal{M}]_{ij} = \int_\Omega \boldsymbol{W}_i^{(1)}.\boldsymbol{W}_j^{(1)} d\Omega$ and $[\mathcal{M}_f]_{ij} = \int_\Omega \boldsymbol{W}_i^{(2)}.\boldsymbol{W}_j^{(2)} d\Omega$ are mass matrices and $[C]$ is the incidence matrix. The permittivity, as well as permeability, can be described in the Laplace domain as follows in a general form

$$\varepsilon(s) = \sum_{n=0}^{p} a_n s^n \Big/ \sum_{n=0}^{p} b_n s^n \tag{2}$$

The appropriate bilinear transformation consistent with the CN method

$$\frac{\partial g(t,\mathbf{r})}{\partial t} = f(t,\mathbf{r}) \xrightarrow{CN} \frac{g^n(\mathbf{r}) - g^{n-1}(\mathbf{r})}{\Delta t} = \frac{f^n(\mathbf{r}) + f^{n-1}(\mathbf{r})}{2} \tag{3}$$

can be easily obtained by applying z-transform to it as

$$s \mapsto \frac{2}{\Delta t}\frac{1-z^{-1}}{1+z^{-1}} \tag{4}$$

Substituting this transformation into (2), the permittivity can be obtained as

$$\varepsilon(z) = \frac{c_0 + c_1 z^{-1} + ... + c_p z^{-p}}{1 + d_1 z^{-1} + ... + d_p z^{-p}} \tag{5}$$

in the z-domain. Where the unknown coefficients c_i and d_i are related to a_n, b_n and Δt. The difference form of the electric field constitutive relation (1b) can be reached using (5) as

$$\{d\}^n = [\mathcal{M}^\dagger]\{e\}^n + \{w\}^{n-1} \tag{6}$$

where $[\mathcal{M}^\dagger] = c_0[\mathcal{M}]$ and $\{w\}^n$ depends on the previous values of $\{d\}^n$ and $\{e\}^n$.

Following a similar procedure, the update equation for the magnetic field constitutive relation in a dispersive media can be written as

$$\{h\}^n = [\mathcal{M}_f^\dagger]\{b\}^n + \{g\}^{n-1} \tag{7}$$

On the other hand, applying the CN method to the semi-discretized Maxwell equations (1a) results in

$$\frac{\{d\}^{n+1} - \{d\}^n}{\Delta t} = [C]^T \frac{\{h\}^{n+1} + \{h\}^n}{2} \tag{8a}$$

$$\frac{\{b\}^{n+1} - \{b\}^n}{\Delta t} = -[C] \frac{\{e\}^{n+1} + \{e\}^n}{2} \tag{8b}$$

Unconditional stability of these equations, known as the CN-FETD, have not been proved yet. However, numerical simulations do not show any instability problem [10]. Substituting (6) and (7) into (8) and eliminating variables related to the magnetic field, one can obtain

$$\begin{aligned}&\left\{[\mathcal{M}^\dagger] + \frac{(\Delta t)^2}{4}[\mathcal{S}^\dagger]\right\}\{e\}^{n+1} = \\ &2\left\{[\mathcal{M}^\dagger] - \frac{(\Delta t)^2}{4}[\mathcal{S}^\dagger]\right\}\{e\}^n - \left\{[\mathcal{M}^\dagger] + \frac{(\Delta t)^2}{4}[\mathcal{S}^\dagger]\right\}\{e\}^{n-1} \\ &- \left\{\{w\}^n - 2\{w\}^{n-1} + \{w\}^{n-2}\right\} + \frac{\Delta t}{2}[C]^T\left\{\{g\}^n - \{g\}^{n-2}\right\}\end{aligned} \tag{9}$$

where $[\mathcal{S}^\dagger] = [C]^{\mathrm{T}}[\mathcal{M}_f^\dagger][C]$ is the conventional stiffness matrix. In a non-dispersive media, $\{g\}$ and $\{w\}$ vectors vanish. Hence, (9) reduces to the second-order vector wave equation discretized by the Newmark-β with $\beta = 1/4$, which is proved to be US; hence, we can conclude that the CN-FETD is the mixed form of it.

III. Results

To validate the formulation described earlier, calculation of the reflection (Γ) and transmission (T) coefficients (normal incidence) of a 5cm wide dielectric slab with the following permittivity and permeability model has been considered

$$\varepsilon(s) = \varepsilon_\infty + \frac{\sigma_e}{s} + G_{e_1}\frac{(\varepsilon_s - \varepsilon_\infty)\,\omega_{e_1}^2}{s^2 + 2\delta_{e_1}s + \omega_{e_1}^2} + G_{e_2}\frac{(\varepsilon_s - \varepsilon_\infty)\,\omega_{e_2}^2}{s^2 + 2\delta_{e_2}s + \omega_{e_2}^2} \tag{10}$$

with the following parameters

$$\begin{gathered}\sigma_e = 0, G_{e_1} = 3, G_{e_2} = 5, \delta_{e_1} = 0.1\omega_{e_1}, \delta_{e_2} = 0.1\omega_{e_2}, \\ \varepsilon_s = 5\varepsilon_0, \varepsilon_\infty = 3.6\varepsilon_0, \omega_{e_1} = 3\pi \times 10^9, \omega_{e_2} = 2 \times 10^9\end{gathered}$$

$$\begin{gathered}\sigma_m = 0, G_{m_1} = 0.3, G_{m_2} = 0.5, \delta_{m_1} = 8 \times 10^{-8}\omega_{m_1}, \\ \delta_{m_2} = 2 \times 10^{-4}\omega_{m_2}, \mu_s = 2.5\mu_0, \mu_\infty = 1.5\mu_0, \\ \omega_{m_1} = 4\pi \times 10^9, \omega_{m_2} = \pi \times 10^9\end{gathered}$$

The time-step is roughly four times the stability limit of the leap-frog method; but, no sign of instability has been observed after 8 million iterations. Fig. 1 shows reflection and transmission coefficients along with the exact solution. Although this media has very sharp resonances, the numerical solution shows an excellent agreement with the exact one.

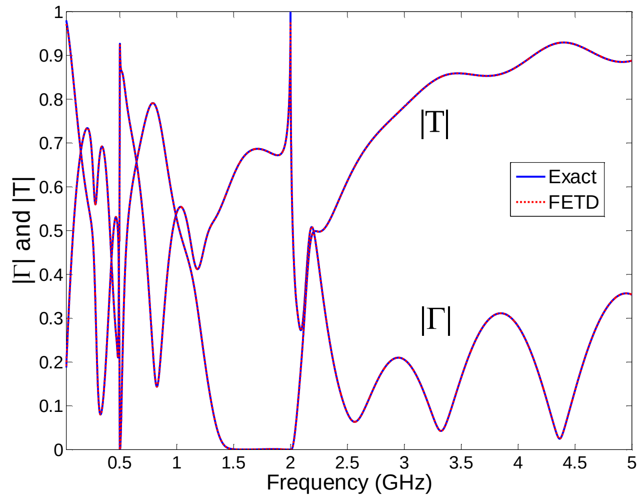

Figure 1: Numerical solution of the reflection (Γ) and transmission (T) coefficients along with the exact solution.

IV. Conclusion

We have described a FETD approach to solve arbitrary linear dispersive media. Using the bilinear transformation method has made the method more flexible than the convolution-based methods. It has been also shown that CN-FETD is actually equivalent to the US vector wave equation if Whitney elements are utilized in spatial discretization.

Full details and additional formulations and results arising from this approach will be presented at the conference.

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