# Dual Discrete Geometric Methods in Terms of Scalar Potential on Unstructured Mesh in Electrostatics

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***Abstract*—The dual formulations established on the dual unstructured meshes using the discrete geometric method for electrostatic field problems are presented. The formulations are both in terms of scalar potential. Compared to the traditional dual formulation in terms of vector potential, the proposed method is more efficient with reduced number of unknowns and computational complexity. The positive definiteness of the constitutive matrices requires the dual meshes satisfy the Voronoi-Delaunay condition. For the elements not satisfying this condition, a modification of the centre node is proposed. The complementary bounds of electrical energy are studied through a micro-electro-mechanicals-systems (MEMS) comb driver example. A comparison between the DGM and the finite element method (FEM) is performed.**

***Index Terms*—Computational electromagnetics, electrostatics, dual formulations, discrete geometric method.**

## I. Introduction

On the framework of differential geometry, various numerical methods such as the finite element method (FEM), the finite integral method (FIT) and the cell method (CM), can be unified with the geometrical description. They differ only by the Hodge discretization of constitutive laws. The analogy between the various methods has been addressed in [1]. The discrete geometric method (DGM) has become an attractive alternative in computational electromagnetics in the past decade [1]-[6].

It is well known that the dual formulations have the advantage of providing complementary energy bounds [7]. One of the benefits is to get global quantities such as the impedances with better accuracy and reduced computational cost. In the case of electrostatic field using the FEM, the dual formulation works with the vector potential and uses the edge elements for the discretisation. It requires in addition, the generation of links between regions (cavities) that contain charges [8]. The complexity of the vector formulation prevents its wide exploration. The dual formulation in terms of vector potential has been explored recently using discrete geometric approach [9]. The computational complexity remains elevated. It has been pointed out in [6] that the dual formulation of electrostatic field can be built through the dual mesh using the scalar potential, but the study was limited to the orthogonal hexahedral mesh. As the majority of practical engineering applications requiring fine analysis uses unstructured meshes, we propose in this paper to extend the dual formulation on dual mesh to unstructured (tetrahedral) mesh. Compared to the vector potential formulation, the proposed method reduces the number of unknowns and the computational complexity.

The discretisation of the Hodge operator, i.e. the computation of the constitutive matrix is carried out through the division of length and facet of respective mutually orthogonal primal and dual meshes [10], which results in a diagonal constitutive matrix. To have the constitutive matrix positive definite, the discrete grid should be constrained by Voronoi-Delaunay condition, i.e. the circumcenter of the tetrahedron should resides in the interior of the element. For the bad shaped elements that this condition is not satisfied, a modification of the centre node is proposed.

## II. Dual Formulations on Dual Meshes

For a typical electrostatic field problem, the primary and dual formulations in terms of scalar potential are readily derivable using DGM:

$$\text{Primal:} \quad \mathrm{G}^{\mathrm{T}} \mathrm{M}_{\varepsilon}^{\mathrm{p}} \mathrm{G}\, \bar{\mathrm{v}}^{\mathrm{p}} = \bar{\rho}^{\mathrm{d}}, \tag{1}$$

$$\text{Dual:} \quad \mathrm{D}\, \mathrm{M}_{\varepsilon}^{\mathrm{d}} \mathrm{D}^{\mathrm{T}}\, \bar{\mathrm{v}}^{\mathrm{d}} = \bar{\rho}^{\mathrm{p}}, \tag{2}$$

where G and D are incident matrices that map, respectively, the nodal degrees of freedom (DoF) to edge DoF and the facet DoF to volume DoF on the primary mesh $\mathcal{M}_{\mathrm{p}}$, while the transposes $-\mathrm{D}^{\mathrm{T}}$ and $-\mathrm{G}^{\mathrm{T}}$ are their counterparts on the dual mesh $\mathcal{M}_{\mathrm{d}}$. $\mathrm{M}_{\varepsilon}^{\mathrm{p}}$ and $\mathrm{M}_{\varepsilon}^{\mathrm{d}}$ represent the constitutive matrices which are discrete versions of Hodge operator, $\bar{\rho}^{\mathrm{p}}$ and $\bar{\rho}^{\mathrm{d}}$ are volume charges, and $\bar{\mathrm{v}}^{\mathrm{p}}$ and $\bar{\mathrm{v}}^{\mathrm{d}}$ are unknown nodal potential vectors. The superscripts p and d indicate that they belong, respectively, to the primal mesh $\mathcal{M}_{\mathrm{p}}$ or the dual mesh $\mathcal{M}_{\mathrm{d}}$. Equations (1) and (2) are to be solved subject to the Dirichlet boundary condition. The degrees of freedom (DoF) of the primal formulation (1) is the number of nodes of $\mathcal{M}_{\mathrm{p}}$ excluding Dirichlet nodes and the DoF of the dual formulation (2) is the number of elements of $\mathcal{M}_{\mathrm{p}}$, i.e. the number of nodes of the dual mesh $\mathcal{M}_{\mathrm{d}}$. Compared to the dual formulation in terms of vector potential, the formulation (2) has fewer unknowns and in addition, there is no need to build the links between the charge regions (or the electrodes).

## III. Computation of Constitutive Matrices

Using the DGM, $\mathrm{M}_{\varepsilon}^{\mathrm{p}}$ and $\mathrm{M}_{\varepsilon}^{\mathrm{d}}$ are diagonal matrices built respectively on the edges of the primal mesh $\mathcal{M}_{\mathrm{p}}$ and on the edges of the dual mesh $\mathcal{M}_{\mathrm{d}}$. Their dimensions are respectively, the number of edges and the number of facets of the mesh $\mathcal{M}_{\mathrm{p}}$. The elements of $\mathrm{M}_{\varepsilon}^{\mathrm{p}}$ are capacitances defined on each edge of $\mathcal{M}_{\mathrm{p}}$. Its ith element is the capacitance on the edge i, given by

$$m^{p}_{\varepsilon ii} = \sum_{e \in E_i} \frac{\varepsilon_e S^{d}_{ei}}{L^{p}_{i}}, \tag{3}$$

where $\varepsilon_e$ is the permittivity of the element e, $E_i$ is the set of n elements containing the edge i of the primal mesh $\mathcal{M}_p$, $L^p_i$ and $S^d_{ei}$ are length and surface determined by the circumcenter of the element as shown in Fig.1 (a). It can be noted that $m^p_{\varepsilon ii}$ represents n elemental capacitances connected in parallel around the edge i.

On the other hand, the elements of $M^d_\varepsilon$ represent capacitances defined on each edge of $\mathcal{M}_d$ , i.e. on each facet of $\mathcal{M}_p$. On the facet j, we have

$$\frac{1}{m^{d}_{\varepsilon jj}} = \sum_{e \in E_j} \frac{L^{d}_{ej}}{\varepsilon_e S^{p}_{j}}, \tag{4}$$

where $E_j$ is the set of 2 elements sharing the facet j of the mesh $\mathcal{M}_p$, $L^d_{ej}$ and $S^p_j$ are length and surface determined by the circumcenter as shown in Fig.1 (b). The element $m^d_{\varepsilon jj}$ is in fact the two elemental capacitances across the facet j in series.

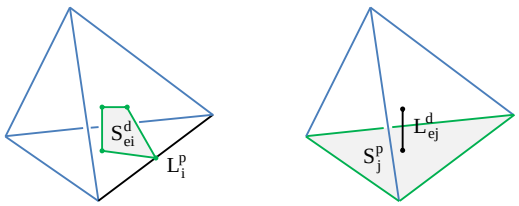


Fig. 1. Elemental contribution to constitutive matrices:
(a). For the computation of $m^p_{\varepsilon ii}$ , (b). for the computation of $m^d_{\varepsilon ii}$

The positive definiteness of matrices $M^p_\varepsilon$ and $M^d_\varepsilon$ requires that the circumceter of the tetrahedron dropped inside of the element (Voronoi-Delaunay mesh). However this is not all the case due to bad shaped elements constrained by the complicate structures. To avoid negative values of $m^p_{\varepsilon ii}$ and $m^d_{\varepsilon ii}$, we propose to shift the centre point (the node of the dual mesh $\mathcal{M}_d$) for the bad chapped elements.

## IV. Example of Validation

To validate the dual DGM on unstructured mesh and to investigate energy bounds of both FEM and DGM dual formulations, we consider a MEMS comb driver and compute the capacitance between the two sets of comb fingers, as shown in Fig. 2. The two conductors are placed in the air and above an infinite ground plane. The computation domain is discretized by tetrahedral mesh.

To compute the total and coupling capacitances, 1-volt voltage is applied on the conductor 1 and 0-volt voltage on the conductor 2 and the ground plane. The problem is solved with the DGM and the FEM using the same mesh. The capacitance values calculated with a tetrahedral mesh including 17171 nodes and 98551 elements are listed in Table I. We observe that the results of the DGM are quite close to those of the FEM. The complementary energy bounds and the energy convergence along with the mesh refinement for both FEM and DGM will be reported in the full paper.

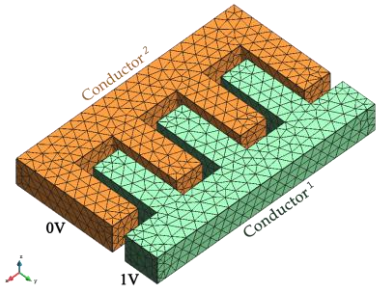


Fig. 2. An electrostatic MEMS comb driver (the infinite ground plane and the air are set invisible).

TABLE I
Capacitance Results with Different Dual Methods (unit: *f*F)

| Methods | | Number of unknowns | Total capacitance | Coupling capacitance |
|---|---|---|---|---|
| FEM | Primal | 9185 | 5.133 | 3.861 |
| | Dual | 79015 | 4.459 | 3.245 |
| | Average | - | 4.796 | 3.553 |
| DGM | Primal | 9185 | 5.182 | 3.875 |
| | Dual | 64066 | 4.464 | 3.252 |
| | Average | - | 4.823 | 3.564 |

## V. Conclusion

A dual DGM on unstructured mesh in terms of scalar potential in electrostatics is proposed. Compared to the vector potential formulation, the method provides the similar results but with reduced computational complexity. The complementary energy bounds are demonstrated as the case with the FEM, the study supports the idea in a step further that complementarity is related to the geometry discretization.

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