Multilevel Preconditioning for Time-harmonic Eddy Current Problems Solved with Hierarchical Finite Elements

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***Abstract*—A multilevel preconditioner is described for high order finite element analysis of eddy currents using the $\mathbf{T}-\Omega$ formulation. The preconditioner combines *p*-type multiplicative Schwartz for the higher order degrees of freedom and auxiliary space preconditioning for the lowest order. Algebraic multigrid is applied to the first order scalar spaces. The new preconditioner performs 4 times faster than a standard preconditioner on a test problem with 13.3 million unknowns.**

***Index Terms*—finite element methods, eddy currents, multigrid methods.**

I. INTRODUCTION

Three-dimensional eddy current problems treated with the finite element method (FEM) often lead to very large matrices. Generally these are solved with Krylov subspace iteration with a preconditioner that greatly affects the speed of convergence. The most common preconditioners are based on incomplete LU factorization (ILU) or on symmetric successive over-relaxation (SSOR). On the other hand, multigrid preconditioners have been very successful in many fields of engineering, particularly with scalar problems. True multigrid requires a series of nested meshes, difficult to arrange in practical problems. Algebraic multigrid (AMG) overcomes this limitation. AMG for scalar potentials is well established [1]. A similar method has been applied to matrices arising from Whitney edge elements [2] [3], but for this case an alternative approach seems promising: Auxiliary Space Preconditioning (ASP) [4] [5]. This is a way of transforming the vector problem into a series of scalar problems, to which AMG is well suited. In [6] ASP was applied to an eddy current formulation using the magnetic vector potential.

In this paper we apply ASP to the $\mathbf{T}-\Omega$ formulation. Furthermore, we consider the case that high-order, hierarchical elements [7] are used and supplement ASP with a multi-level technique called *p*-type multiplicative Schwartz (*p*MUS) [8].

II. THE $\mathbf{T}-\Omega$ METHOD

The phasor magnetic field, $\mathbf{H}$, solves the following equations in conducting (V_c) and non-conducting (V_o) regions:

$$\nabla\times\boldsymbol{\sigma}^{-1}\cdot\nabla\times\mathbf{H}+j\omega\boldsymbol{\mu}\cdot\mathbf{H}=0 \qquad \text{in } V_c \tag{1a}$$

$$\nabla\cdot\boldsymbol{\mu}\cdot\mathbf{H}=0 \qquad \text{in } V_o \tag{1b}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the tensor conductivity and permeability, respectively. In the $\mathbf{T}-\Omega$ method, these equations are solved by representing $\mathbf{H}$ as $\mathbf{H}_s-\nabla\Omega$ in V_o, where $\mathbf{H}_s$ is a precomputed source field, and as $\mathbf{T}-\nabla\Omega$ in V_c, where $\mathbf{T}$ is an unknown vector potential. The tetrahedral finite elements in [7] are used, including the vertex functions. The rotational functions represent $\mathbf{T}$ and the gradient functions represent $\nabla\Omega$. The decomposition is gauged by building a tree from the edges of the tetrahedra in the conductors; rotational functions for tree edges are omitted. Application of the FEM leads to a matrix equation $Ax=b$.

III. AUXILIARY SPACE PRECONDITIONING

The basis functions of the order 1 element are classified as "first order", the functions added to take an order 1 element to order 2 are classified as "second order", etc. All the first order basis functions in the mesh are numbered first, then the second order functions, etc. This leads to the following partitioning:

$$A=\begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix} \tag{2}$$

where p is the highest order present in the mesh.

The preconditioner is a V-cycle that approximately solves $Ax=r$. The cycle consists of a series of steps, each of which computes an improvement, Δx, in x, and reduces r accordingly. In the following, the x and r updates are implied.

First apply pMUS. Solve $A_{pp}\Delta x_p=r_p$ by a single step of backward Gauss-Seidel (BGS). Repeat with $A_{p-1,p-1}$, then with $A_{p-2,p-2}$, etc. After A_{11}, switch to ASP. Transfer the residual to four auxiliary spaces: piecewise linear scalars, N, and the three Cartesian components of the space, N^3, of *nodal* vector fields. Now, for each space, use scalar AMG. Apply BGS, transfer the residual to an algebraically coarser problem, apply BGS again, etc. When the matrix problem is small enough, solve it exactly by a direct method. That is the descending half of the V-cycle. The ascending half is the mirror image. Forward Gauss-Seidel (FGS) is used instead of BGS. A W-cycle preconditioner may also be constructed.

IV. NUMERICAL RESULTS

To demonstrate the performance of proposed method, the simulation of a hollow, conducting cube placed in a uniform field is considered. The geometry and the mesh are shown in Fig.1. The permeability and the conductivity of the conductor are $\mu=\mu_0$ and $\sigma=1.45\times10^6$ S/m respectively. The skin

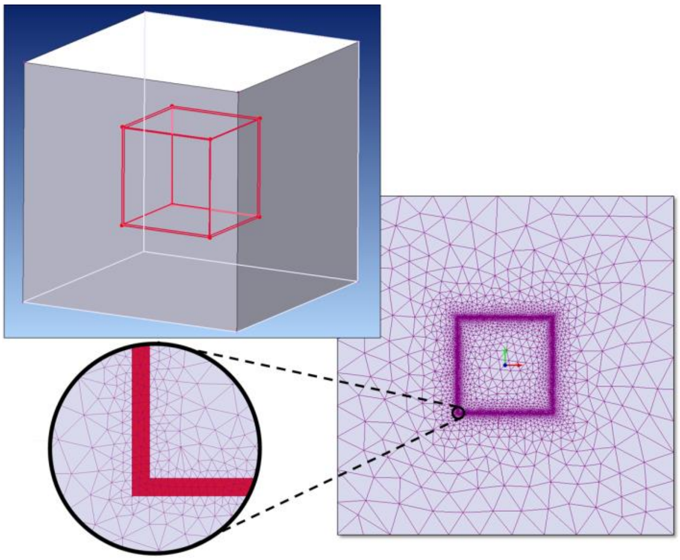

Fig 1. Illustration of geometry and discretization for the conducting cube problem.

depth at the excitation frequency, 100 Hz, is $\delta = 42.8$ mm. The length of the cube is 7m and the wall thickness is 2δ. The computational domain, V, is a cube of side 20m. The source field $\mathbf{H}_s$ is zero in this example, but the scalar potential is constrained on two opposite surfaces of V to produce a magnetic field that would be uniform in the absence of the conducting cube. The domain is discretized with 3,750,322 tetrahedra, resulting in 663,471 scalar and 2,212,397 vector unknowns at the lowest order. Three steps of *p*-adaption are applied. The matrix problems are solved by the preconditioned COCG method [9], with the termination criterion set to 10^{-6}.

Table I presents the number of unknowns, the number of COCG iterations and the corresponding CPU times for solving the matrix problem at each adaptive step. *p*MUS preconditioners with different treatments at the first order are applied. Selected methods for approximating A_{11} are ILU with 0 level of fill-in, SSOR, and ASP with V- or W- cycles. For the ASP method, scalar AMG employs a hierarchy of three levels with 517,421, 167,589 and 55,560 unknowns.

The comparison shows the superior iteration count and run time of ASP at each adaptive step. The experiments also indicate that a significant improvement can be achieved for higher order systems if a better algebraic solver is used at the first order. The cumulative CPU time for solving the problem

TABLE I
SOLUTION DETAILS FOR THE CONDUCTING CUBE PROBLEM

	Step 1		Step 2		Step 3	
Unknowns	2,875,868		8,578,954		13,345,009	
Nonzeros	66,707,982		350,799,470		615,703,547	
A_{11} Prec.	Iter.	CPU Time (hh:mm:ss)	Iter.	CPU Time (hh:mm:ss)	Iter.	CPU Time (hh:mm:ss)
ILU(0)	741	00:28:13	685	01:57:35	628	02:51:45
SSOR	608	00:23:49	652	01:41:55	654	03:03:14
ASP (V-Cycle)	**150**	**00:12:28**	**205**	**00:39:09**	**206**	**01:08:05**
ASP (W-Cycle)	**60**	**00:13:23**	**88**	**00:28:19**	**94**	**00:45:55**

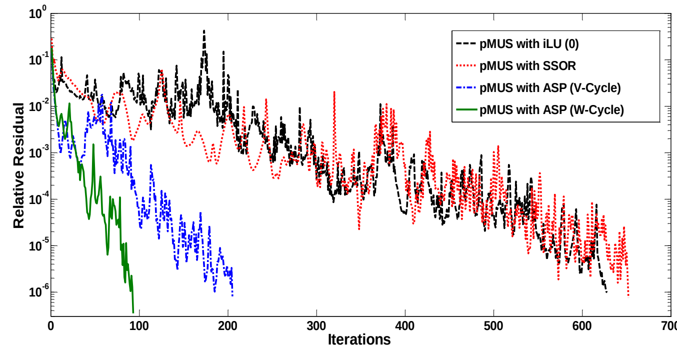

Fig. 2. Convergence history of COCG with different preconditioners for solving the conducting cube problem at *p*-adaptive step 3.

with three *p*-adaptive steps using *p*MUS and ASP (W-cycle) is reduced by about 73% compared to ILU(0), and by about 71% compared with SSOR.

The convergence history of COCG at step 3 is shown in Fig 2. It is observed that while the convergence rate for ILU and SSOR are nearly same, the ASP (W-cycle) is able to reduce the number of iterations by a factor of almost 7.

V. CONCLUSION

The *p*MUS multilevel method combined with ASP is an effective technique for solving the large matrices that arise when using hierarchical finite elements to solve time-harmonic eddy current problems.

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