Trefftz-discontinuous Galerkin and finite element multi-solver technique for modeling time-harmonic EM problems with high-conductivity regions

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***Abstract*—This paper introduces a multi-solver technique in order to enhance the broadband performance of time-harmonic finite element methods when solving electromagnetic problems with high-conductivity regions. The high-conductivity regions are modeled by the Trefftz-discontinuous Galerkin (TDG) formulation while the FEM is used in the rest of the computational domain. The novel multi-solver technique couples the methods by making use of Robin-type transmission conditions. The efficiency is illustrated by computing the broadband frequency sweep of the inductance of a two-wire transmission line: the necessary number of DoFs in the multi-solver case decreases significantly compared to the pure FE case if the two solutions are generated with the same error criterion.**

***Index Terms*—Maxwell problem, Multi-solver technique, Finite element method, Trefftz-discontinuous Galerkin method.**

I. Background

Vector finite element (FE) methods are extremely successful tools for solving time-harmonic electromagnetic problems, especially when complex geometry and different materials including dielectrics and conductors are involved [1]. However, many problems contain highly dissipative metallic computational domains where the broadband FE modeling is quite inefficient even with utilizing high-order elements, and the mesh size can grow prohibitively high with increasing frequency. To mitigate this problem, modern FE solvers may introduce different surface or volumetric models of the conductive regions for different frequency bands. This can lead to certain inconsistencies in broadband frequency sweeps and increased complexity in the algorithm.

Recently we have experimented with the Trefftz-discontinuous Galerkin (TDG) method for solving time-harmonic electromagnetic (EM) problems using different T-complete bases [2][3]. Our conclusion is that the method gave superior performance in highly dissipative mediums compared to the FE method. This experience inspired us to try a multi-solver approach to enhance the performance of adaptive FE simulators when solving broadband electromagnetic problems with metallic computational regions. This paper introduces an efficient multi-solver technique for such problems by utilizing the TDG method in the high-conductivity regions and the FE method elsewhere. To couple the two computational domains, the technique makes use of Robin transmission conditions similarly to the modern domain decomposition (DD) formulations [4][5][6].

II. Model Definition

Let $\Omega \subset \mathbb{R}^3$ be a bounded, polyhedral domain and $\mathbf{n}$ the unit normal vector field on $\partial\Omega$. For the sake of brevity, we consider the following computational model with impedance boundary condition in terms of the electric field, $\mathbf{E}$:

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{E}\right) - \omega^2 \varepsilon \mathbf{E} = \mathbf{0} \qquad \text{in } \Omega \tag{1}$$

$$\left(\mu^{-1} \nabla \times \mathbf{E}\right) \times \mathbf{n} + j\omega\eta^{-1}\mathbf{E}_T = \mathbf{g} \qquad \text{on } \partial\Omega \tag{2}$$

where ε, μ and η are piece-wise constant, and $\mathbf{g} \in L_T^2(\partial\Omega)$. Anisotropic ε and μ are allowed except in the good conductors where the TDG computational model is used. (TDG can be generalized to anisotropic materials too but such cases seem to have little practical importance in high-conductivity conductors.) The discussions in the rest of the paper can trivially be extended to include most other boundary conditions or excitations.

III. Trefftz-Discontinuous Galerkin Method

The framework of the discontinuous Galerkin (DG) method for time-harmonic Maxwell problems is described in [7], based on which we can derive the TDG method. See the details of the derivation, discretization and error analysis in [3] and [8].

The fundamental discretized TDG weak form of problem (1) and (2) for all $\partial K \in \mathcal{F}_h$ is

$$\int_{\partial K} \mathbf{n} \times \hat{\mathbf{E}}_h \cdot \overline{\left(\mu^{-1} \nabla \times \mathbf{w}_h\right)} dS - j\omega \int_{\partial K} \mathbf{n} \times \hat{\mathbf{H}}_h \cdot \overline{\mathbf{w}}_h dS = 0 \tag{3}$$

where mesh $\mathcal{T}_h$ is a finite element partition of Ω with mesh width of h, $K \in \mathcal{T}_h$ denotes a mesh element, and $\mathcal{F}_h = \bigcup_{K \in \mathcal{T}_h} \partial K$ is the skeleton of the mesh $\mathcal{T}_h$. In TDG the weighting functions $\mathbf{w}_h$ have the Trefftz property, that is, they satisfy $\nabla \times \left(\mu^{-1} \nabla \times \mathbf{w}_h\right) - \omega^2 \overline{\varepsilon} \mathbf{w}_h = \mathbf{0}$ or more precisely they are *T-complete* [3]. (The bar denotes complex conjugate.) We define the *numerical traces* (or “numerical fluxes”) in (3) as

$$\hat{\mathbf{E}}_h = \left\{\mathbf{E}_h\right\} - j\omega^{-1}\beta \left[\!\left[\mu^{-1} \nabla_h \times \mathbf{E}_h \right]\!\right]_T \tag{4}$$

$$\hat{\mathbf{H}}_h = j\omega^{-1} \left\{ \mu^{-1} \nabla_h \times \mathbf{E}_h \right\} + \alpha \left[\!\left[\mathbf{E}_h \right]\!\right]_T \tag{5}$$

where we use the standard DG notation for the average, $\{\mathbf{G}\} = \frac{1}{2}\left(\mathbf{G}^{+} + \mathbf{G}^{-}\right)$, and the jump, $[\![\mathbf{G}]\!]_{T} = \mathbf{n}^{+} \times \mathbf{G}^{+} + \mathbf{n}^{-} \times \mathbf{G}^{-}$, on $\partial K^{-} \cap \partial K^{+}$ of neighboring elements K^{+} and K^{-}. $\mathbf{E}_{h}$ represents the discretization of $\mathbf{E}$ by *T-complete* approximation functions and the symbol $\nabla_{h} \times (\cdot)$ denotes the element-wise application of $\nabla \times (\cdot)$. Note that the inhomogeneous boundary condition (2) is also incorporated in (4) and (5) when ∂K is on $\partial\Omega$. Finally, we want to point out that the solution of (3)–(5) also incorporates the satisfaction of Robin transmission conditions on the mesh skeleton.

IV. Multi-Solver Technique

The nonoverlaping DD formalisms utilize Robin transmission conditions to "glue" the field approximations of the different domains together [4][5][6]. Since the field continuity at element interfaces is ensured through enforcing Robin transmission conditions in TDG (3)–(5) too, it is quite straightforward to couple the TDG and FE domains efficiently by a scheme similar to the DD techniques. We will show the details of the coupled multi-solver system in the full version of the paper.

V. Results

A. Verification

Our implementation of the TDG – FE coupled technique is successfully verified by matching numerical field solutions to the analytical solution of a standard eddy current benchmark problem where a cylindrical coil generates eddy currents in an infinite metal plate. (The coil axis is perpendicular to the plate.) This benchmark arrangement is widely used in eddy-current NDE (Nondestructive Evaluation) because it has a semi-analytical full-wave solution that can be generated by high accuracy [9].

B. Two-Wire Transmission Line

In order to illustrate the performance of the multi-solver technique, we investigate the broadband behavior of a two-wire transmission line system depicted in Figure 1(a). The transmission line is excited by ports and we calculate the frequency sweep of the inductance as the output. Figure 1(b) illustrates the skin and proximity effects at increasing frequencies.

Figure 2(a) compares the analytical frequency sweep of the inductance to the frequency sweeps calculated by the pure FE and multi-solver options. The inductances are calculated with the same accuracy criterion in the two numerical solutions. Figure 2(b) plots the DoF growth for the two numerical solutions. Note that the FE solutions for the last two frequency points are missing because the solver ran out of memory on our workstation due to the excessive mesh size increase in the conductors during the adaptive refinement process that assures consistent accuracy.

Nonetheless, we can see strong indications that the multi-solver approach is much more efficient than the pure FE solver for broadband problems including high-conductivity regions.

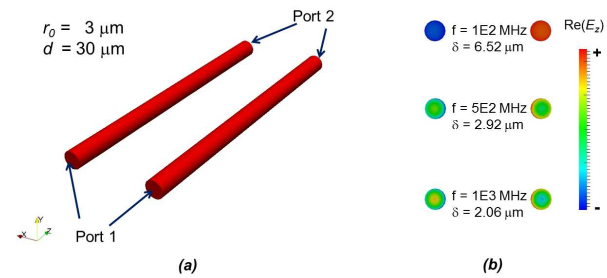

Fig. 1. *(a)* Two-wire transmission line problem excited by Port 1 and 2, and *(b)* the distribution of the longitudinal electric field component in the conductors at different frequencies. δ is the skin depth.

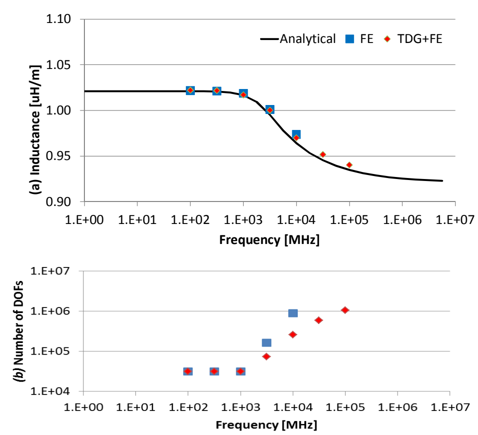

Fig. 2. Two-wire transmission line problem: *(a)* Inductance and *(b)* number of DoFs vs. frequency. All computations have been carried out on a workstation, having 32 GB memory and a quad-core Intel Xeon X3440 CPU.

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