A Time-Domain Discontinuous Galerkin Trefftz Method

Fritz Kretzschmar*, Sascha M. Schnepp†, Igor Tsukerman‡, and Thomas Weiland§
*Graduate School of Computational Engineering,
Technische Universitaet Darmstadt, Dolivostraße 15, 64293 Darmstadt, Germany
kretzschmar@gsc.tu-darmstadt.de
†Laboratory for Electromagnetic Fields and Microwave Electronics,
ETH Zurich, Gloriastraße 35, 8092 Zurich, Switzerland
schnepps@ethz.ch
‡Department of Electrical & Computer Engineering,
The University of Akron, Akron, Ohio 44325-3904, USA
itsukerman@uakron.edu
§Institut fuer Theorie Elektromagnetischer Felder,
Technische Universitaet Darmstadt, Schlossgartenstrasse 8, 64289 Darmstadt, Germany
weiland@temf.tu-darmstadt.de

***Abstract*—In this article we present a novel Discontinuous Galerkin Finite Element Method for wave propagation problems. The method employs space-time Trefftz basis functions that satisfy the underlying partial differential equations exactly in an element-wise fashion. We demonstrate spectral convergence of the approximation error in the L_2-norm. High order time integration is an inherent property of the method, and we obtain spectral convergence in the entire space-time domain of interest (i.e. in space and time simultaneously), which is not achieved by any common approaches of time-stepping finite element discretizations of space only.**

***Index Terms*—Computational Electromagnetics, Maxwell equations, Finite Element Methods, Convergence of numerical methods, Discontinuous Galerkin Method, Trefftz Method**

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Discontinuous Galerkin Finite Element Methods (DG-FEM) [1]–[3] are a major class of tools to numerically simulate complicated electromagnetic systems. In this article we present a highly accurate type of DG-FEM for time-domain applications. A distinguishing new attribute of the method is the use of Trefftz basis functions [4]–[6], which, by definition, satisfy the underlying partial differential equations exactly in an element-wise fashion. The method is, hence, a Discontinuous Galerkin Trefftz Finite Element Method (DGT-FEM). Since Trefftz functions are required to solve the equations exactly, they need to depend both on space and time. Therefore, we obtain a space-time DGT method. We obtain high order time integration in a consistent manner, and achieve spectral convergence in space-time. Existing numerical methods exhibit polynomial convergence or, at best, spectral convergence in space only.

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For a wave propagating in a given direction x with one-component fields $E \equiv E_y$ and $H \equiv H_z$, we can write the system of sourceless Maxwell's equations in a coordinate free form

$$\nabla^{\mathrm{T}} \cdot \eta_\mu \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla^{\mathrm{T}} \cdot \eta_\epsilon \cdot \mathbf{F} = 0. \tag{1}$$

Here, $\mathbf{F} \equiv (E, H)^{\mathrm{T}}$ is the field vector defined in the space-time domain of interest Ω. The matrices η_ϵ and η_μ represent the material operators, $\nabla = (\partial_t, \partial_x)^{\mathrm{T}}$ is the differential operator. After this equation is obtained we apply the standard testing procedure with test functions $\mathbf{v} \equiv (v^E, v^H)^{\mathrm{T}}$ and subsequently perform an integration by parts yielding

$$\begin{aligned}\int_{\partial\Omega} v^E(\eta_\epsilon \mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}\Omega + \int_{\partial\Omega} v^H(\eta_\mu \mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}\Omega \\ - (\nabla^{\mathrm{T}} \cdot v^E, \eta_\epsilon \mathbf{F})_\Omega - (\nabla^{\mathrm{T}} \cdot v^H, \eta_\mu \mathbf{F})_\Omega = 0,\end{aligned} \tag{2}$$

where $\mathbf{n} = (n_x, n_t)^{\mathrm{T}}$ is the unit outward normal on the space-time domain boundary $\partial\Omega$. A numerical implementation of 2 is described in the following section.

A T F

Trefftz basis functions are a set of problem specific solutions of the underlying partial differential equations. Trefftz functions in the form of plane waves have been previously applied in the frequency domain in the context of a DG method for the homogeneous Helmholtz equation [7]–[9]. In this work, however, a space-time-domain Trefftz method is developed–to our knowledge, for the first time.

For (1+1)-dimensional Maxwell's equations (i.e. Maxwell's equations in one spatial and one temporal dimension) transport polynomials of the form

$$\mathbf{u}^{p,\pm} = \begin{pmatrix} u^{E,p,\pm} \\ \\ u^{H,p,\pm} \end{pmatrix} = \begin{pmatrix} \pm(x \mp v\,t)^p \\ \\ \frac{1}{Z}(x \mp v\,t)^p \end{pmatrix}, \tag{3}$$

provide a Trefftz basis. Here the first and second component, u^E and u^H, of any basis function in (3) represents the electric and magnetic field, respectively. We would like to emphasize that the material parameters $Z = \sqrt{\mu/\epsilon}$ and $v = 1/\sqrt{\mu\epsilon}$ (i.e. intrinsic impedance and local speed of light respectively) enter the basis functions directly. Each order is included twice for

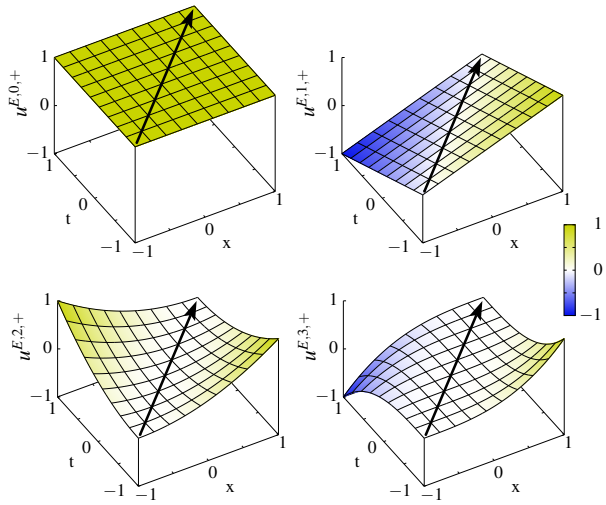

Figure 1: Transport polynomials of orders $p = 0, 1, 2, 3$ in free space (i.e. $\epsilon_{\mathbf{r}} = 1$ and $\mu_{\mathbf{r}} = 1$) to describe waves traveling rightwards.

representing the directions of the transverse electromagnetic waves. A wave is traveling rightwards if E and H have equal sign, and leftwards otherwise. The first four transport polynomials in free space are depicted in Fig. 1.

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To date, we have simulated different wave propagation scenarios. First, we verified the propagation of a Gaussian wave in free space. Then, we applied the method to a Gaussian wave propagation through several material interfaces, comforming the boundaries of the computational cells. Finally, we used DGT to simulate Gaussian wave transmission and reflection at a material interface located inside a computational cell. Remarkably, as seen in Fig. 2, the presence of the material interface does not affect the accuracy and the spectral convergence ot the method. This is a key feature unique to the proposed approach.

In Fig. 3 the relative L_2-error obtained with the DGT method (boxes) is compared to the error obtained using a centered DG discretization of space combined with a Leapfrog scheme [10] (triangles). The straight line in the semi-logarithmic plot demonstrates spectral convergence of DGT-FEM, whereas the Leapfrog DG scheme is limited by the second-order time integrator.

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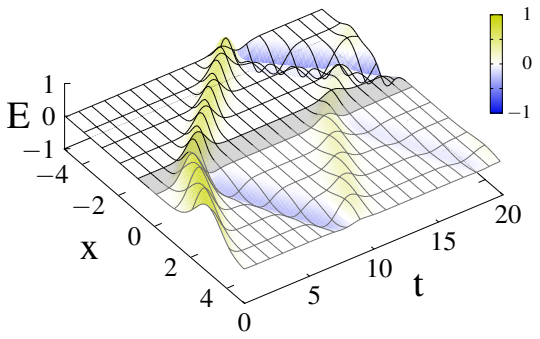

Figure 2: Electric field of a one-dimensional Gaussian wave with a medium interface at $x_0 = -0.25$, simulated with the DGT-FEM. The material interface is located in the interior of an element. The solution in the whole space-time domain of interest $(x, t) \in [-5, 5] \times [0, 20]$ is displayed.

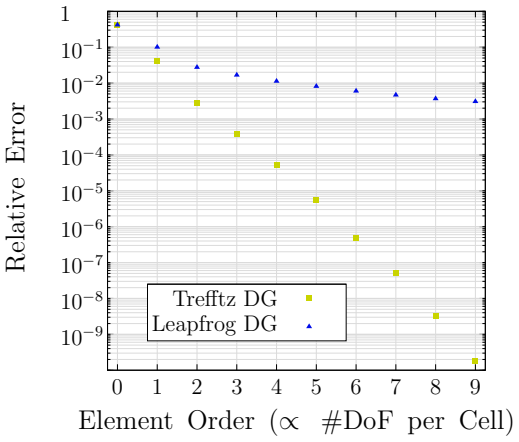

Figure 3: Order dependent L_2 error of DGT and Leapfrog DG.

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