Computational Performances of Natural Element and Finite Element Methods

Yves Marechal, Brahim Ramdane and Diego Bothelo Pereira
G2Elab, Grenoble Electrical Engineering Laboratory
11 rue des mathématiques, Saint Martin d'Hères, 38402, France
Yves.Marechal@grenoble-inp.fr

***Abstract*—this paper compares the numerical performance of 2 numerical methods, the finite element method and the natural element method. Natural element method is relatively recent and is based on functions belonging to the Voronoï cell family. Whereas it is has been proved that this method gives smoother and more accurate solutions than the finite elements, its computational cost is also known as being higher. In this paper we compare computational efficiency, i.e. accuracy for a given cost, of finite elements and natural elements, for both Lagrange and Sibson shape functions. We also bring into the comparison a Voronoï cell based finite difference scheme which proves to be very efficient. The error is calculated using dual formulations.**

***Index Terms*— Finite element methods, finite difference methods, Natural element methods, computational efficiency**

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I. Introduction

Meshless methods have been widely studied during the past 20 years [1]. They proved to bring high accuracy solutions, but have also inherent difficulties to handle properly boundary conditions, material jump, and connection with finite elements [2]. In the late 90s, a new method called Natural Element Method (NEM) appears [3] [4], based on the Voronoï cell surrounding each node. This method proves to overcome all major drawbacks of previous meshless methods, while keeping the smooth and highly accurate solutions observed with the latter. This method has also been successfully used in the field of electrical engineering problems [6].

Still the computational cost of Natural Element Method is significantly higher than the usual cost of Finite Elements. In this work, we propose to compare computational efficiency - i.e. accuracy for a given cost, of finite elements and natural elements - of both FEM and NEM, for which we take into account both Lagrange and Sibson functions. Moreover, a finite difference implementation of Voronoï cell functions (NFDM) is also included in the comparison.

II. Voronoi Cell Functions

A. Voronoï Diagram

The natural element method uses the concept of Voronoï cell. Let consider a set of nodes $N = \{n_1, n_2, n_3, \dots, n_N\}$ distributed in the whole domain. The Voronoï diagram is a subdivision of the domain into cells, where each cell T_i associated to node n_i is such that any point in T_i is closer to node n_i than to any other node n_j for $i \neq j$. These cells are the so called Voronoï cells.

Considering for instance a 2D space, T_i is the region of the plane that contains the points $\boldsymbol{x}$ closest to node n_j than to any other node as shown in Fig. 1. The Delaunay triangulation, which is the dual of the Voronoï diagram, is built by connecting the nodes whose Voronoï cells have common boundaries, thus leading to the notion of natural neighbors.

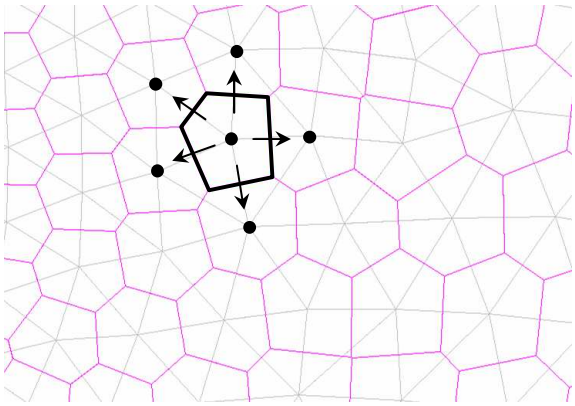

Fig. 1. A Voronoï diagram (pink color) and its associated Delaunay triangulation (gray color). One node, its Voronoï cell and its 5 natural neighbors are highlighted.

B. Shape Function

Based on the Voronoï diagram, a natural element shape function can be expressed. In the literature, various formulas are proposed. Among the most used are the Sibson and Lagrange functions [4].

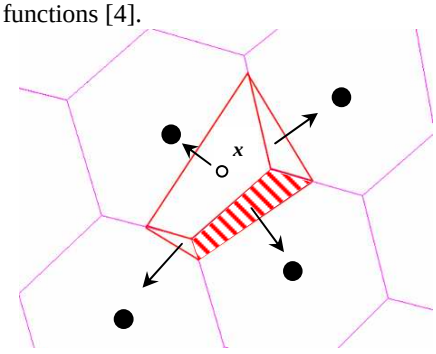

Fig. 2. NEM shape function computation.

For the Sibson functions, their determination may be performed by analogy with the classical FE shape functions. For first-order triangles for instance, it is well known that the shape functions are given by a ratio of surfaces. For Sibson shape functions, the same principle is applied to the Voronoï cells as shown in Fig. 2. The shape function at a point $\boldsymbol{x}$ is thus given by (2) where each $S_i(x)$ represents the sub area of the Voronoï cell centered on $\boldsymbol{x}$ linked to the natural neighbor n_i , illustrated by the hashed region in Fig. 2.

$$\Phi_i(x) = \frac{S_i(x)}{\sum_j S_j(x)} \qquad (2)$$

It can be noticed that when moving to 3D, Sibson function will involve relatively complex calculations of volumes.

In 2D (resp. 3D), Lagrange functions are based on a ratio of length (resp. surface), and therefore are more cost efficient. However, continuity property in that case is slightly deteriorated.

C. *Finite Difference and Finite Element with Voronoï Cell Functions*

Both collocation for NFDM and Galerkin schemes for NEM can be built using Voronoï cell functions. Finite element approach requires the integration of the functions, and due to the rational aspect of these functions and their complex geometric support, a relatively large number of Gauss points is generally used. On the contrary, the finite difference approach described in [5] is very efficient in terms computation time and maintains good accuracy even on unstructured grids.

III. Numerical Results and Conclusions

To illustrate the computational performance of each method, we are presenting numerical results based on an electrostatic problem describing an L shaped capacitance. This test problem has been solved by all methods (FEM, Laplace and Sibson NEM, and NFDM) and for each case, by dual formulations to determine an error estimator.

First, Fig. 4 plots the relative computation time. As we may expect, NFDM performs best since no integration is needed together with 2nd order FEM, then comes 1st order FEM and last NEM which pays the cost of both numerical integration and a larger band size.

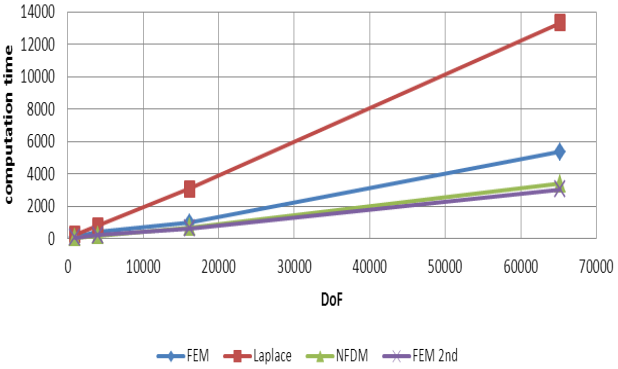

Fig. 4. Computation time for NEM (Laplace), FEM (1st and 2nd order) and NFDM according to the number of degree of freedom.

Then, regarding accuracy, Fig. 5 clearly demonstrates the very interesting accuracy of NEM. But it also interesting to notice that the finite difference method based on Voronoï cell functions is also performing much better than first order finite elements.

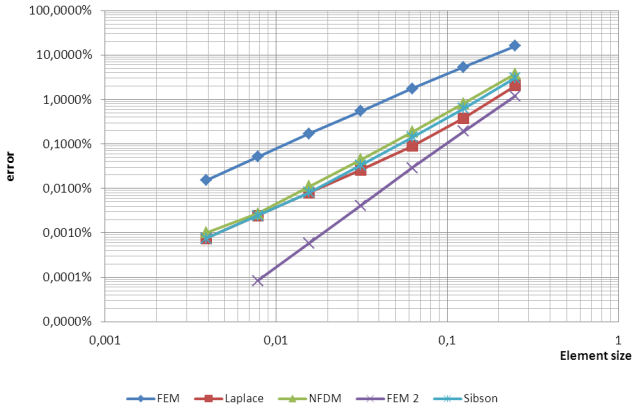

Fig. 5. Accuracy of NEM, FEM (1st and 2nd order) and NFDM according to the length between nodes.

Last, Fig. 6 compares the computational performance of these 3 methods. For a given accuracy, the methods based on Voronoï cell functions - NFDM and NEM - are significantly much faster than 1st order FEM (more or less one order of magnitude in time). This tends to clearly demonstrate the very interesting behavior of the Voronoï cell approach: the interpolation has a high degree of continuity, it does not exhibit edge noise, and last, it has a kind of isotropic interpolation behavior around a node, very similar to the one observed with meshless methods. A more comprehensive review of these results will be developed in the extended version of the paper.

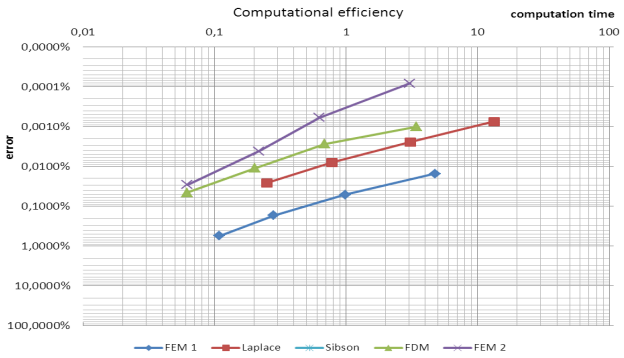

Fig. 6. Compared performance of NEM, FEM and NFDM.

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