Efficient Solution of Transient Eddy Current Problems Using Linear/Nonlinear Domain Substructuring

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***Abstract*—In this paper the domain substructuring approach is adapted to the nonlinear transient eddy current problem such that the linear model parts are treated more efficiently. A matrix decomposition of the linear subproblem is executed on before hand and its factorization is used through out the simulation. For a general 3D problem these non-sparse factors are replaced by sparse approximations in order to increase the efficiency of the solution process. Numerical results of a test example validate this approach.**

I. Introduction

The space-discretization of a magnetoquasistatic field problem yields a nonlinear differential-algebraic system, [1]. Implicit time-integration is inevitable and thus large linear systems must be solved in every Newton-Raphson iteration. Efficient solvers for linear systems have a complexity in the order of the degrees of freedom. Further improvements require parallelization and a reduction of the number of unknowns. To this end, we propose a substructuring method based on Schur complements, [2]. This is especially efficient for the curl-curl equation because large subdomains are linear (e.g. air) and furthermore those domains are described by a static system.

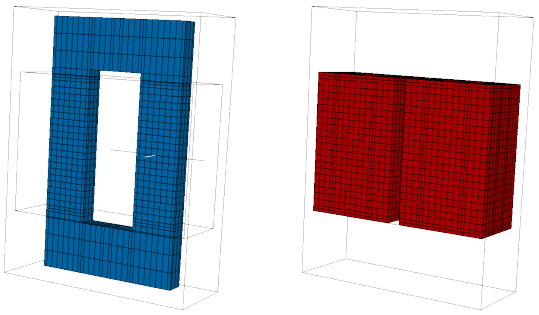
Here the substructuring procedure isolates the linear equations and solves them only once. This yields an (sparse approximate) factorization. In either case, this knowledge is used during the time-integration.

In this paper we discuss methods for the efficient computation of the Schur complement, the matrix-vector multiplications and suitable preconditioners for the iterative method, e.g. by using sparse approximations of the inverse, [3].

II. Problem Formulation

Let us consider a nonlinear magnetoquasistatic problem, e.g. the transformer, Fig. 1. The excitation within such a model is given by a source current, which is imposed in a conductor region. Typically, this region is a coil consisting of thin strands, such that no local eddy currents are present. For such a configuration the space-discrete MQS-problem, obtained from edge elements, can be decomposed as follows

$$\mathbf{M}\dot{\mathbf{a}}_\kappa + \mathbf{K}_\kappa(\mathbf{a}_\kappa)\mathbf{a}_\kappa + \mathbf{K}_\Gamma\mathbf{a}_0 \;=\; 0, \tag{1a}$$



(a) Iron core (b) Copper coils (stranded conductors)

Fig. 1. Transformer model: iron core (blue) exhibits eddy currents and a nonlinear permeability. The coils (red) are modeled linearly without eddy currents (the strands are below skin-depth). The surrounding air (white) is linear and non-conductive.

$$\mathbf{K}_\Gamma^\top\mathbf{a}_\kappa + \mathbf{K}_0\mathbf{a}_0 \;=\; \mathbf{X}_0\boldsymbol{\imath}. \tag{1b}$$

The conductivity matrix $\mathbf{M}$ models the eddy currents. The degrees of freedom (DoFs) of the line-integrated vector potential are split into variables that are directly affected by eddy currents $\mathbf{a}_\kappa$ and the rest $\mathbf{a}_0$. The curl-curl matrix is decomposed accordingly into a nonlinear part $\mathbf{k}_\kappa := \mathbf{K}_\kappa(\mathbf{a}_\kappa)\mathbf{a}_\kappa$, which models magnetic saturation, and a linear part $\mathbf{K}_0$. We will assume $\mathbf{K}_0$ to be gauged, e.g., by adding a grad-div regularization [4], such that $\mathbf{K}_0$ is positive definite. This gauging yields a matrix, which corresponds to a vector Laplacian. Both operators are coupled by at the interface matrix $\mathbf{K}_\Gamma$.

Before we discuss the source term, we equip system (1a–1b) with a coupling equation (for the electric network coupling):

$$\mathbf{X}_0^\top\dot{\mathbf{a}}_0 + \mathbf{R}_0\boldsymbol{\imath} \;=\; \mathbf{v}. \tag{1c}$$

Matrix $\mathbf{R}_0$ describes the DC resistances of the conductors. The circuit coupling matrix $\mathbf{X}_0$ distributes lumped currents. Each column corresponds to a conductor (e.g. coil), which is contacted externally. The structure in (1) assumes that these conductors do not exhibit eddy currents and are made of linear materials.

Now, the elimination of $\mathbf{a}_0$ yields the Schur system

$$\mathbf{M}\dot{\mathbf{a}}_\kappa + \Big(\mathbf{K}_\kappa(\mathbf{a}_\kappa) - \mathbf{K}_\Gamma\mathbf{K}_0^{-1}\mathbf{K}_\Gamma^\top\Big)\mathbf{a}_\kappa = \mathbf{X}_\kappa\boldsymbol{\imath}, \tag{2a}$$

$$\mathbf{X}_\kappa^\top \dot{\mathbf{a}}_\kappa + \mathbf{L}_0 \dot{\boldsymbol{\imath}} + \mathbf{R}_0 \boldsymbol{\imath} = \mathbf{v} \qquad (2b)$$

with the new coupling matrix and lumped inductance matrix:

$$\mathbf{X}_\kappa := -\mathbf{K}_\Gamma \mathbf{K}_0^{-1} \mathbf{X}_0 \quad \text{and} \quad \mathbf{L}_0 := \mathbf{X}_0^\top \mathbf{K}_0^{-1} \mathbf{X}_0. \qquad (3)$$

The matrix $\mathbf{X}_\kappa$ maps the excitations into the nonlinear, eddy current domain and $\mathbf{L}_0$ describes the linear magnetic coupling in the eddy current-free domain.

In this formulation the evaluation of the vector potential $\mathbf{a}_0$ in the inner part of the eliminated subdomain is never necessary: neither for the time-stepping procedure (that requires only initial values for the differential components $\mathbf{a}_\kappa$), nor for the evaluation of the current/voltage relation (2b). Thus the common forward/backward substitutions known from domain decomposition can be avoided. This is of course at the price of the 'denser' Schur-complement matrix in (2).

III. Algorithm and Complexity

Time-discretization of (2) yields a nonlinear symmetric problem, that is solved by Newton-Raphson. The resulting linear system has the benefit of an improved (effective) condition number compared to (1). The spread in the eigenvalues is reduced, because we have removed an essential material jump in the problem, [5]. Furthermore the number of unknowns is reduced. This promises better convergence properties when using an iterative solver, e.g. preconditioned conjugate gradients (PCG), [6], in the Newton-Raphson iteration.

Now, the matrix $\mathbf{K}_\Gamma \mathbf{K}_0^{-1} \mathbf{K}_\Gamma^\top$ must be solved repeatedly for given right-hand-sides. Typically a LU-decomposition returns sparse factors only for 2D problems, but a decomposition of the curl-curl matrix for a 3D problem will yield rather dense factors. This will increase the computational costs.

Assuming the number of DoFs along one spatial direction is $\mathcal{O}(n)$, the DoFs in the 3D model scales with $\mathcal{O}(n^3)$. An optimal multigrid solver returns a solution in linear computational time ($\mathcal{O}(n^3)$). The number of DoFs at any surface scales with $\mathcal{O}(n^2)$. Hence a matrix representation of an operator on a surface contains $\mathcal{O}(n^4)$ entries (full matrix). Asymptotically, the application of $\mathbf{K}_\Gamma \mathbf{K}_0^{-1} \mathbf{K}_\Gamma^\top$ would dominate the computational cost and would make the method worse than a standard multigrid technique. If a standard LU-factorization for $\mathbf{K}_0^{-1}$ is applied (and computed in a pre-processing step), already the forward/backward substitution will have the same complexity $\mathcal{O}(n^4)$. But applying sparse approximations for this inverse (factorization) can reduce the complexity considerably, e.g. hierarchical matrix techniques, as far as $\mathcal{O}(n^2 \log n^2)$, [7] which is clearly below $\mathcal{O}(n^3)$.

IV. Numerical Test

We simulated the transformer model depicted in Fig. 1. It was discretized by the finite integration technique using 139,995 DoFs. For all numerical tests the PCG method was used with Jacobi preconditioner and the same tolerance $1\mathrm{e}{-4}$.

The convergence study of PCG is shown in Fig. 2 for different settings: a) original formulation (1) without gauging (red line), b) grad-div gauging [4] (green); c) Schur (2) based on grad-div gauging (blue) d) Schur (2) based on grad-div gauging with reduced fill-ins (identical to blue line).

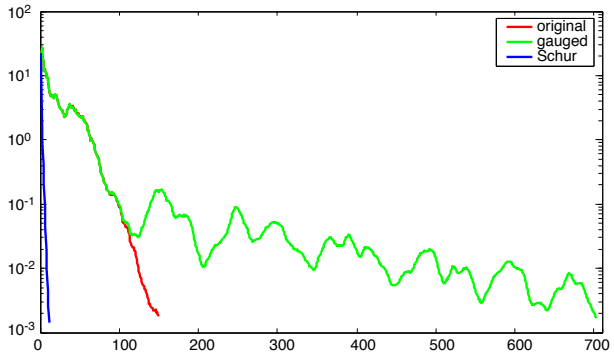


Fig. 2. Convergence plot of PCG for the original formulation with (green) and without gauging (red) and for the gauged Schur system (2) (blue).

If no gauge is applied, a linear systems stemming from (1) is solved in 4s (red, over 100 iterations). Gauging adds nonzeros and increases the spread of the eigenvalues, thus PCG needs about 10s to solve one system (green).

Solving the smaller system (2) requires significantly less iterations (11, blue). But the cost per iteration is higher due to rather dense factors such that it takes 4s, i.e., as long as for the original system. Reducing additionally the number of fill-ins in the Schur complement, while keeping the same level of accuracy, the solution time was less than 2s (still 11 iterations). This underlines the importance of the reduction of fill-ins.

V. Conclusions and Outlook

We have shown that domain substructuring applied to the transient eddy current problem reduces PCG iterations. The costs of each PCG iteration depend on the sparsity of the factorization of the (linear) subproblems. The numerical example have illustrated that a sparse factorization speeds up the solve by a factor of two. The upcoming paper will discuss further examples and techniques.

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References

[1] I. A. Tsukerman, A. Konrad, G. Bedrosian, and M. V. K. Chari, "A survey of numerical methods for transient eddy current problems," *IEEE Trans Magn*, vol. 29, no. 2, pp. 1711 –1716, March 1993.

[2] L. Giraud, A. Marrocco, and J. C. Rioual, "Iterative versus direct parallel substructuring methods in semiconductor device modelling," *Numerical Linear Algebra with Applications*, vol. 12, no. 1, pp. 33–53, 2005.

[3] W. Hackbusch, "Direct domain decomposition using the hierarchical matrix technique," in *Fourteenth International Conference on Domain Decomposition Methods*, I. Herrera, D. E. Keyes, O. B. Widlund, and R. Yates, Eds., 2003.

[4] M. Clemens and T. Weiland, "Regularization of eddy-current formulations using discrete grad-div operators," *IEEE Trans Magn*, vol. 38, no. 2, pp. 569 – 572, March 2002.

[5] K. Lipnikov and P. Deuflhard, "Domain decomposition with subdomain CCG for material jump elliptic problems," *East-West J. Numer. Anal*, vol. 6, pp. 81–100, 1997.

[6] M. Clemens, S. Schöps, H. De Gersem, and A. Bartel, "Decomposition and regularization of nonlinear anisotropic curl-curl DAEs," presented at IGTE 2010, submitted to COMPEL.

[7] M. Bebendorf, *Hierarchical Matrices*, ser. Lecture Notes in Computational Science and Engineering. Berlin: Springer, 2008, no. 63.